Box 13-2 PROPERTIES OF FOURIER TRANSFORMS

Representing a Function by a Fourier Series

Consider a completely arbitrary function $f(\theta)$, defined in the interval $\theta = -\pi$ to $\theta = \pi$. It is possible to represent this function as an expansion in a series of functions with known properties. Only certain sets of functions are suitable for such an expansion and, in the interval $-\pi$ to π, sines and cosines together constitute such a set:

$$f(\theta) = \sum_{n=0}^{\infty} a_n \cos(n\theta) + a'_n \sin(n\theta)$$

where the index n runs through all positive integers. This expansion is called a Fourier series. The coefficients a_n and a'_n are numbers determined by the properties of $f(\theta)$.

As shown in Box 13-1, sines and cosines can be expressed in terms of complex exponentials. Therefore, the Fourier series just given can instead be written as

$$f(\theta) = \sum_{n=-\infty}^{\infty} b_n e^{in\theta}$$

where the index n now runs through both positive and negative values because these are necessary to describe sines and cosines. The coefficients b_n can be found in a simple way by making use of the following result.

For any two integers n and m,

$$\begin{aligned}\int_{-\pi}^{\pi} e^{in\theta}e^{-im\theta}\,d\theta = \int_{-\pi}^{\pi} e^{i(n-m)\theta} &= [1/i(n-m)](e^{i(n-m)\pi} - e^{-i(n-m)\pi}) \\ &= [2/(n-m)]\sin(n-m)\pi = 0 \qquad \text{if } n \neq m \\ &= 2\pi \qquad \text{if } n = m\end{aligned}$$

where the result for $n = m$ can be proven by expanding the sine expression in a power series. Therefore, to find a particular b_m, one performs the integral

$$(1/2\pi)\int_{-\pi}^{\pi} f(\theta)e^{-im\theta}\,d\theta = (1/2\pi)\int_{-\pi}^{\pi} d\theta \sum_{n=-\infty}^{\infty} b_n e^{in\theta}e^{-im\theta} = b_m$$

Note that the integral is carried out over the entire range of θ over which $f(\theta)$ is defined. It often is convenient to be able to work with an arbitary range $-L/2$ to $L/2$ rather than with $-\pi$ to π. This is accomplished by defining a new variable, $x = L\theta/2\pi$, such that when $\theta = \pi$, then $x = L/2$, and when $\theta = -\pi$, then $x = -L/2$. Incorporating this variable into the above equations, and using the fact that $dx = (L/2\pi)d\theta$, we obtain

$$f(x) = \sum_{n=-\infty}^{\infty} b_n e^{2\pi inx/L}$$

$$b_n = (1/L)\int_{-L/2}^{L/2} e^{-2\pi inx/L} f(x)\,dx$$

Fourier Transforms in One Dimension

The function $f(x)$ is defined at all x, whereas the set of coefficients b_n represents an infinite array of numbers, which must be tabulated. Therefore, it is convenient to find an analog of the Fourier series in which the coefficients b_n are replaced by a function, and the summation is replaced by an integral. This representation is called a Fourier transform when the interval over which the function is defined extends from $-\infty$ to $+\infty$.

We define a new continuous variable, $S = 2\pi n/L$, and a new continuous function $g(S) = Lb_n$. Using these, the equation for b_n is transformed to

$$g(S) = \int_{-\infty}^{\infty} e^{-2\pi i S x} f(x)\, dx \qquad \text{(A)}$$

in the limit as $L \to \infty$. The series expansion for $f(x)$ becomes

$$f(x) = \sum_{n=-\infty}^{\infty} [g(S)/L] e^{2\pi i S x}$$

To replace the sum by an integral, note that the interval ΔS corresponds to $(2\pi/L)\Delta n$ from the definition of S. But $\Delta n = 1$ in the summation, and therefore each increment dS in an integral is equivalent to $2\pi/L$ in the sum. Thus,

$$f(x) = (L/2\pi) \int_{-\infty}^{\infty} [g(S)/L] e^{2\pi i S x}\, dS = (1/2\pi) \int_{-\infty}^{\infty} g(S) e^{2\pi i S x}\, dS \qquad \text{(B)}$$

Equations A and B constitute a pair of Fourier transforms that allow $f(x)$ to be calculated if $g(S)$ is known, and vice versa. They are particularly interesting because the variables x and S have opposite dimensions. For example, if x is distance, then S is reciprocal distance. The factor of $(1/2\pi)$ in equation B often is written instead as $(1/\sqrt{2\pi})$ in front of the integrals in both equations A and B.

Fourier Transforms in Three Dimensions

Suppose the function f is now defined in a Cartesian coordinate system with axes x, y, z. For fixed y and z, the function $f(x, y, z)$ can be expanded in a Fourier series in $e^{2\pi i S_x x}$, and the Fourier transform becomes (by analogy to Equation A)

$$g_{yz}(S_x) = \int_{-\infty}^{\infty} e^{-2\pi i S_x x} f(x, y, z)\, dx$$

This expression, in turn, can be expanded in the function $e^{2\pi i S_y y}$ for fixed z, and finally as a function of $e^{2\pi i S_z z}$. The resulting three-dimensional Fourier transform is

$$g(S_x, S_y, S_z) = \int_{-\infty}^{\infty} dz e^{-2\pi i S_z z} \int_{-\infty}^{\infty} dy e^{-2\pi i S_y y} \int_{-\infty}^{\infty} dx e^{-2\pi i S_x x} f(x, y, z)$$

If we use the vector $\mathbf{S}$ to represent the three variables S_x, S_y, and S_z, and we use $\mathbf{r}$ to represent x, y, and z, then the three-dimensional transform can be written very compactly as

$$g(\mathbf{S}) = \int_{-\infty}^{\infty} d\mathbf{r}\, e^{-2\pi i \mathbf{S}\cdot\mathbf{r}} f(\mathbf{r})$$

Similarly, the analog of Equation B becomes

$$f(\mathbf{r}) = (1/2\pi)^3 \int_{-\infty}^{\infty} d\mathbf{S}\, e^{2\pi i \mathbf{S}\cdot\mathbf{r}} g(\mathbf{S})$$