

# Wavelets and multiresolution representations

*Time meets frequency...*

# Time-Frequency resolution

- Depends on the time-frequency spread of the wavelet atoms

Assuming that  $\psi$  is centred in  $t=0$

## Signal domain

$$f_s(t) = \frac{1}{\sqrt{s}} f(t) \rightarrow \|f_s\|^2 = \|f\|^2$$

$$\sigma_t^2 = \int_{-\infty}^{+\infty} t^2 |\psi(t)|^2 dt$$

$$\int_{-\infty}^{+\infty} (t-u)^2 |\psi_{u,s}(t)|^2 dt = s^2 \sigma_t^2$$

## Fourier domain

$$\eta = \frac{1}{2\pi} \int_0^{+\infty} \omega^2 |\hat{\psi}(\omega)|^2 d\omega$$

$$\hat{\psi}_{u,s}(\omega) = \sqrt{s} \psi(s\omega) e^{-i\omega u} \rightarrow \text{center frequency } \eta/s$$

Energy spread around  $\eta/s$

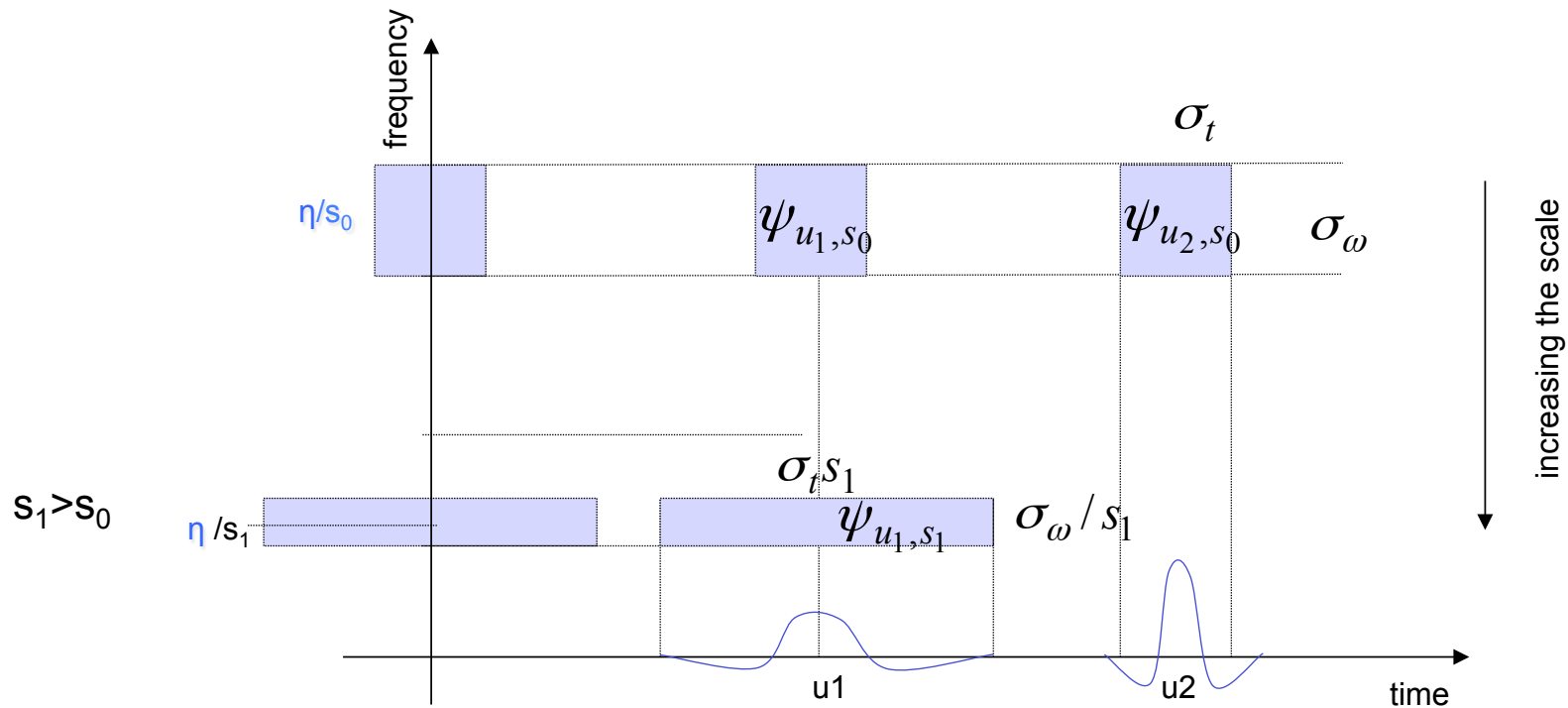
$$\frac{\sigma_\omega^2}{s^2} = \frac{1}{2\pi} \int_0^{+\infty} \left(\omega - \frac{\eta}{s}\right)^2 |\hat{\psi}_{u,s}(\omega)|^2 d\omega$$

## Time/frequency resolution

$$\sigma_{s,t}^2 = s^2 \sigma_t^2$$
$$\sigma_{s,\omega}^2 = \frac{\sigma_\omega^2}{s^2}$$

- The energy spread of a wavelet time-frequency atom corresponds to an **Heisenberg box** centred at  $(u, \eta/s)$  of size  $s\sigma_t$  along the time and  $\sigma_\omega/s$  along the frequency.
- The **area of the rectangle remains equal to  $\sigma_t \sigma_\omega$  at all scales**, while the resolution in time and frequency depends on  $s$ .
- A wavelet defines a **local time-frequency energy density  $P_{wf}$**  which measures the energy in the Heisenberg box of each wavelet centred at  $(u, \eta/s)$ . This energy density is called **scalogram**

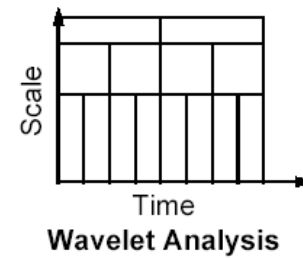
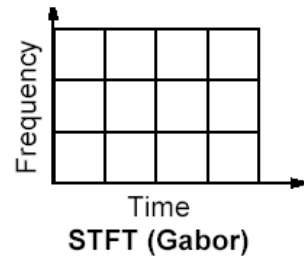
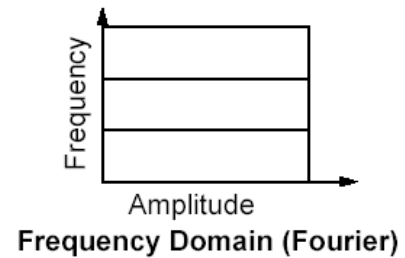
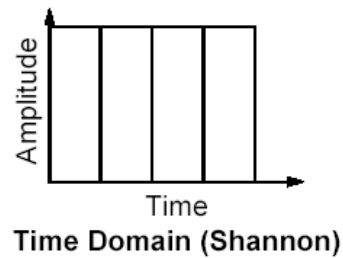
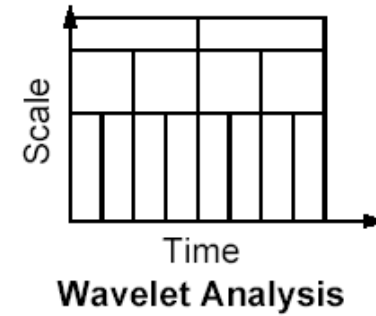
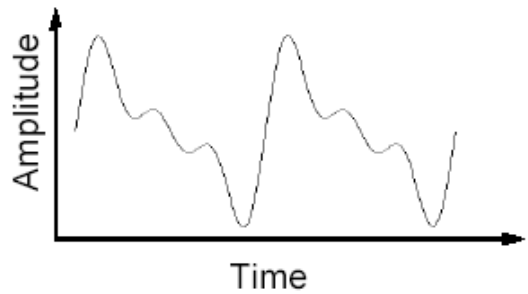
# Time/frequency localization



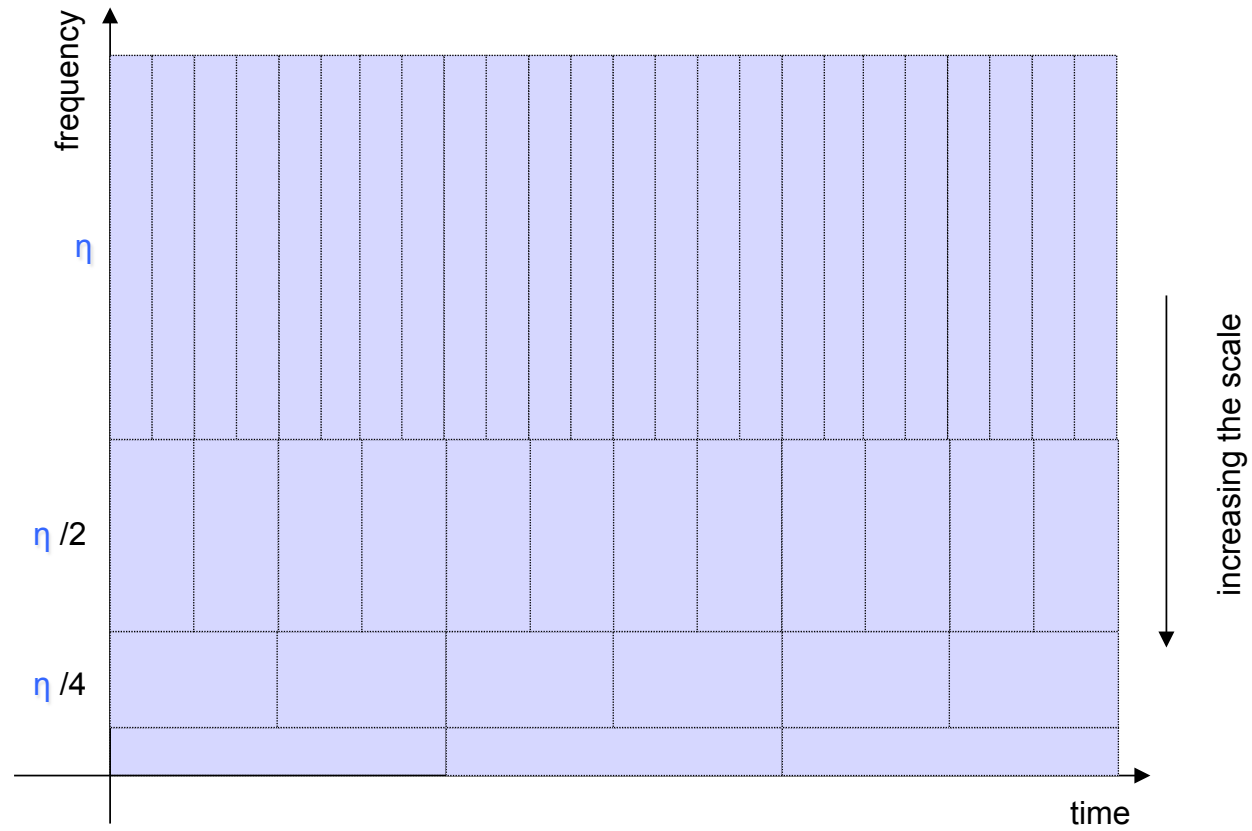
Increasing the scale ( $s$  gets larger) pushes the box towards low frequencies  $\rightarrow$  frequency resolution increases, spatial resolution decreases

**Time spread is proportional to scale**  
**Frequency spread is proportional to 1/scale**

# Wavelet domain



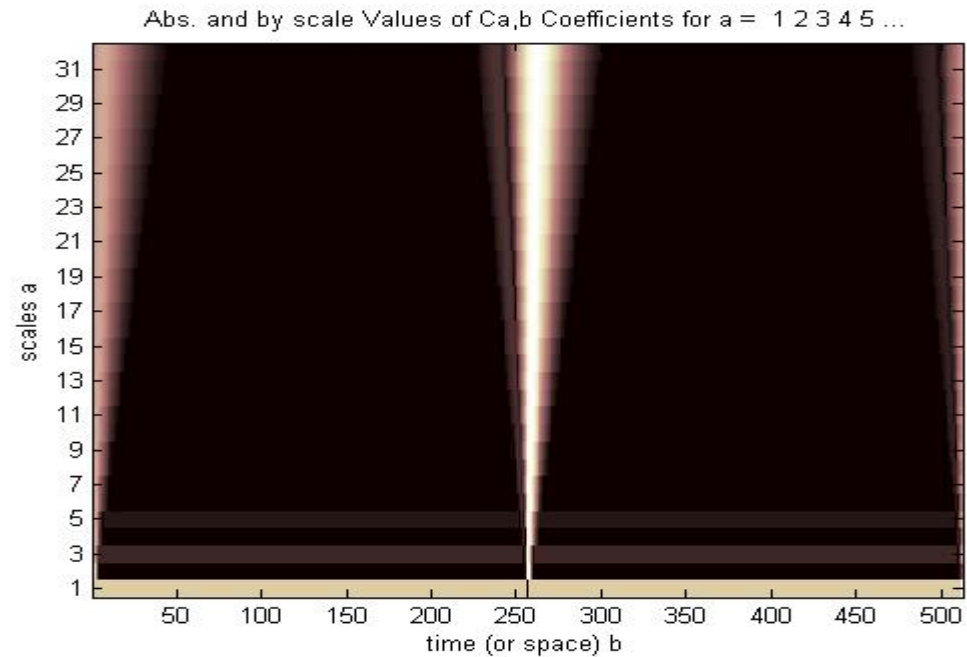
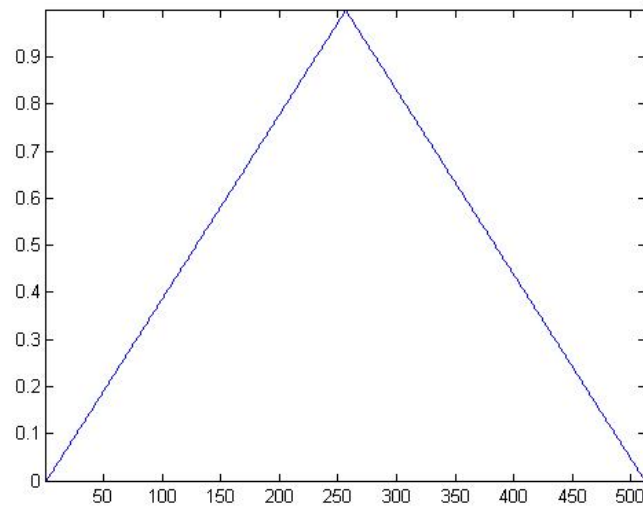
# Dyadic Wavelets



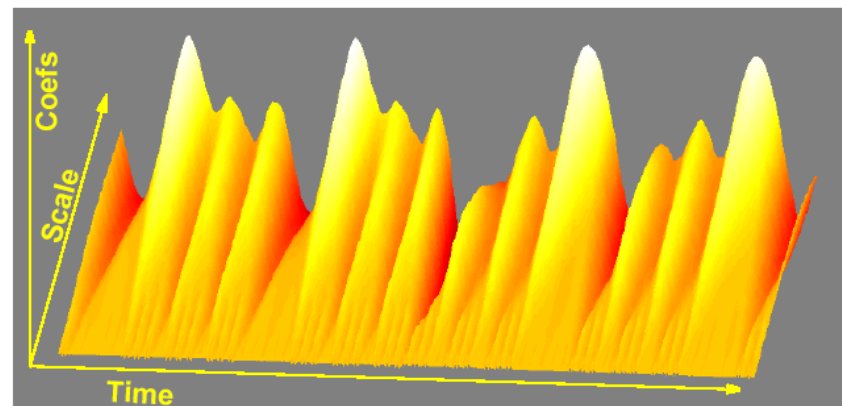
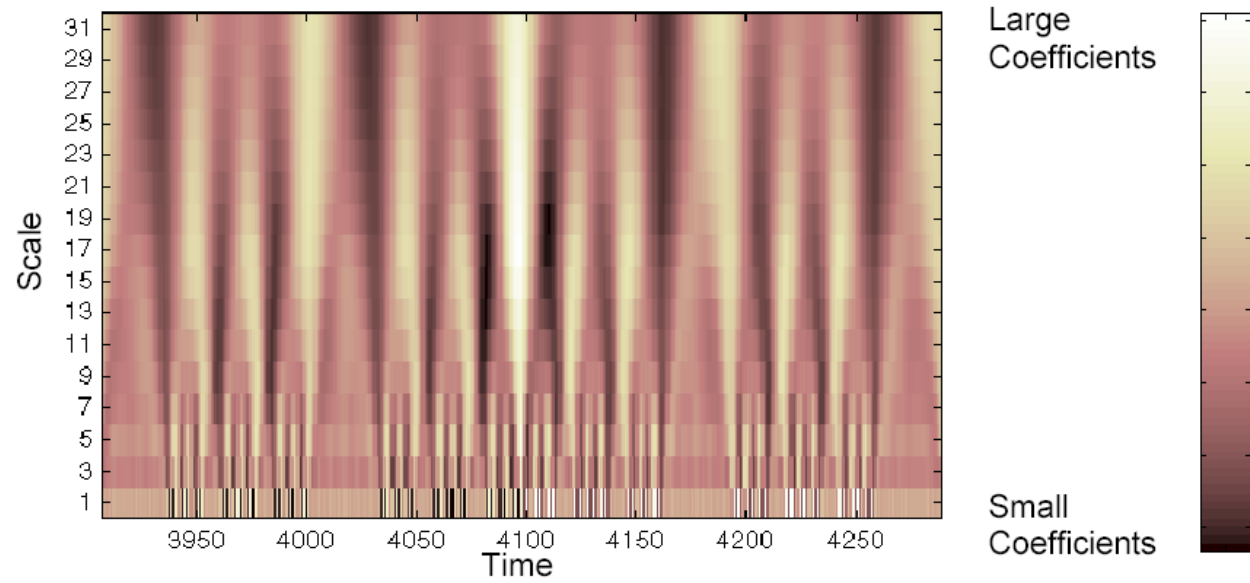
# Scalogram

- The scalogram represents the local time/frequency energy density
  - Energy density in the Heisenberg box of each wavelet  $\psi_{u,s}$

$$P_W f(u, \xi) = |Wf(u, s)|^2 = \left| Wf \left( u, \frac{\eta}{\xi} \right) \right|^2$$

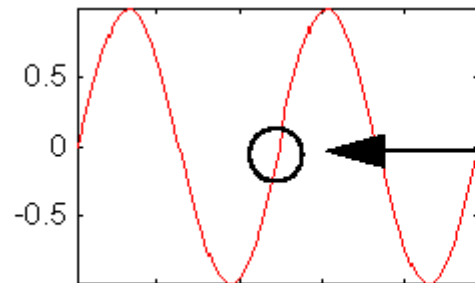


# 3D representation

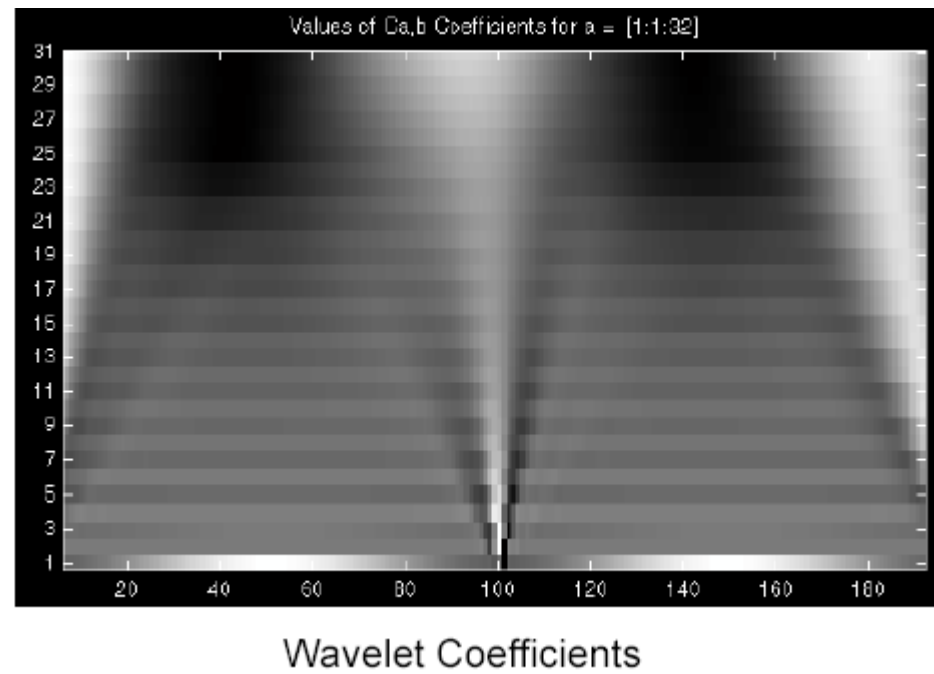
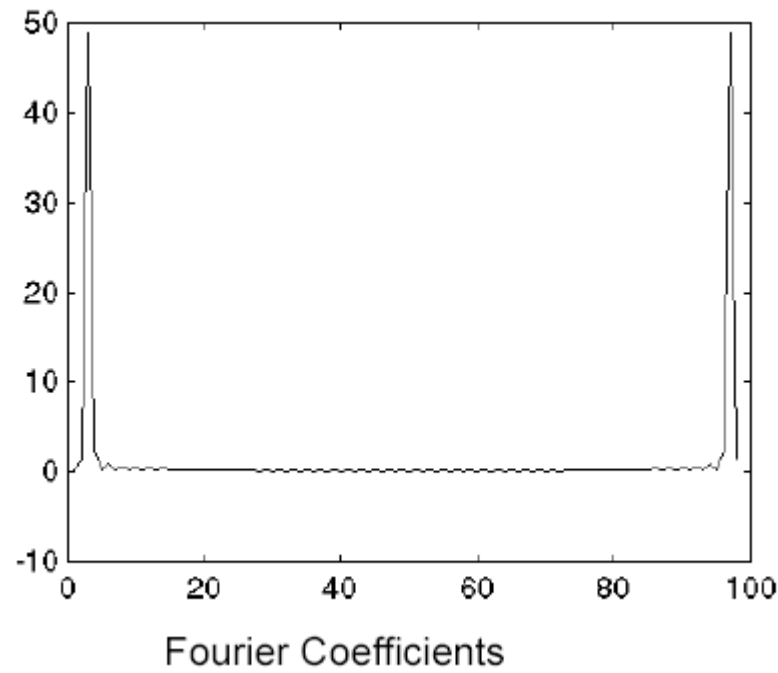




## Local discontinuities



Sinusoid with a small discontinuity



# Real Wavelets

- Detect sharp signal transitions

$$Wf(u, s) = \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) dt$$

- Measures the variations of  $f$  in the neighborhood of  $u$  whose size is proportional to  $s$
- A real WT is complete and maintains energy conservation as long as it satisfies a weak *admissibility condition* (Theorem 4.3, next slide)
- The **decay of the coefficients as  $s$  goes to zero** characterizes the **regularity** of  $f$  in the neighborhood of  $u$

## Real wavelets: Admissibility condition

- Theorem 4.3 (Calderon, Grossman, Morlet)

Let  $\psi$  in  $L^2(\mathbb{R})$  be a real function such that

$$C_\psi = \int_0^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < +\infty$$

Admissibility condition

Any  $f$  in  $L^2(\mathbb{R})$  satisfies

$$f(t) = \frac{1}{C_\psi} \int_0^{+\infty} \int_{-\infty}^{+\infty} Wf(u,s) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) du \frac{ds}{s^2}$$

and

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt = \frac{1}{C_\psi} \int_0^{+\infty} \int_{-\infty}^{+\infty} |Wf(u,s)|^2 du \frac{1}{s^2} ds$$

# Admissibility condition

- Consequences

- The integral is finite if the wavelet has zero average  $\hat{\psi}(0) = 0$ 
  - This condition is nearly sufficient  $\rightarrow$
- If  $\hat{\psi}(0) = 0$  and  $\hat{\psi}(\omega)$  is continuously differentiable, than the admissibility condition is satisfied
  - This happens if it has a sufficient time decay

$$\int_{-\infty}^{+\infty} (1 + |t|) |\psi(t)| dt < +\infty$$

$\rightarrow$  The wavelet function must decay **sufficiently fast** in both time and frequency

## Scaling function (1)

- When  $Wf(u,s)$  is known only for  $s < s_0$ , to recover  $f$  we need a complement of information corresponding to  $Wf(u,s)$  for  $s > s_0$ .
- This is obtained by introducing a *scaling function*  $\phi$  that is an aggregation of wavelets at *scales larger than 1*.
- The modulus of the Fourier transform of  $\phi$  is defined as follows and the complex phase can be arbitrarily chosen

$$|\hat{\phi}(\omega)|^2 = \int_1^{+\infty} |\psi(s\omega)|^2 \frac{ds}{s} = \int_{\omega}^{+\infty} |\psi(\xi)|^2 \frac{d\xi}{\xi}$$

- Remembering that

$$C_{\psi} = \int_0^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < +\infty$$

It appears that

$$C_{\psi} = \lim_{\omega \rightarrow 0} |\hat{\phi}(\omega)|^2$$

## Scaling function (2)

- The scaling function can thus be seen as a *low-pass filter* with *unit gain* ( $\|\phi\|^2 = 1$ )
- Let us denote

$$\phi_s(t) = \frac{1}{\sqrt{s}} \phi\left(\frac{t}{s}\right) \quad \text{and} \quad \bar{\phi}_s(t) = \phi_s^*(-t)$$

- The *low frequency approximation of f at scale s* is

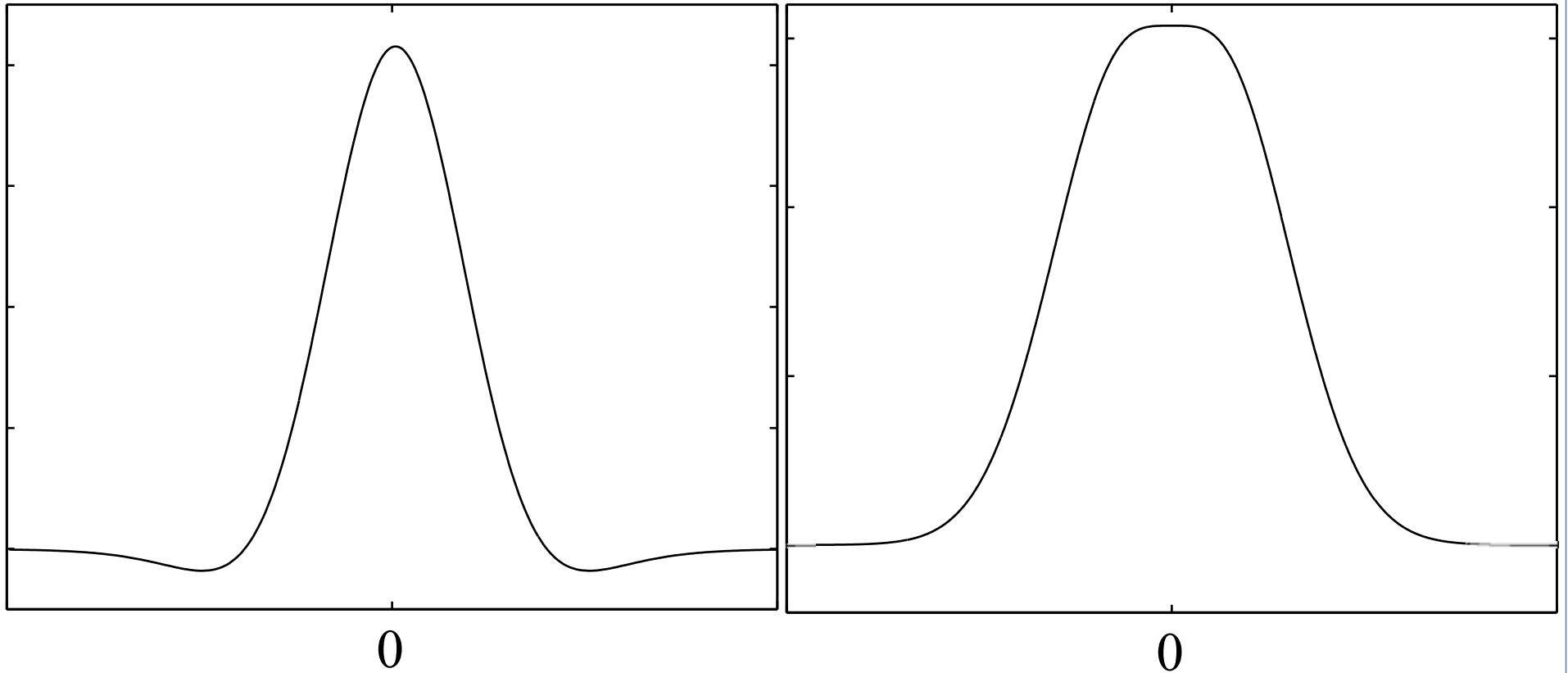
$$Lf(u, s) = \left\langle f(t), \frac{1}{\sqrt{s}} \phi\left(\frac{t-u}{s}\right) \right\rangle = f * \bar{\phi}_s(u)$$

$$\bar{\phi}_s(t) = \phi\left(-\frac{t}{s}\right)$$

# Mexican hat scaling function

$\phi(t)$

$\hat{\phi}(\omega)$



## Wavelet families

$$f(\vec{x}) \leftrightarrow Wf(u, s; \vec{x}) = c_{u,s}(\vec{x})$$

- In general, there is a *redundancy* in the representation
- The *amount* of redundancy depends on the *grids* over which the *u and s* parameters are sampled

*u, s* are real : Continuous WT (CWT, overcomplete representation)

*u* in  $\mathbb{Z}$ ,  $s=a^j$ , *j* in  $\mathbb{Z}$  : Wavelet Frames (DWF, DDWF, overcomplete)

–  $a=2$  Dyadic wavelet frames

$u=k2^j$ ,  $s=2^j$ , *k* in *I* : Discrete Wavelet Transform (DWT) (*critically sampled*)

- Note: removing completely the redundancy leads to complete basis (*critically sampled*)



# Wavelet bases

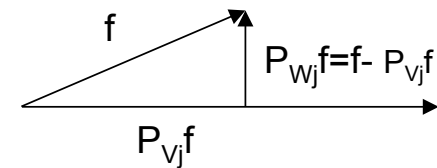
Mallat - Chapter VII

# Wavelet bases

One can construct wavelets such that

$$\left\{ \psi_{j,n}(t) = \frac{1}{\sqrt{2^j}} \psi \left( \frac{t - 2^j n}{2^j} \right) \right\}_{j,n \in \mathbb{Z}^2}$$

is an orthonormal basis for  $L^2(\mathbb{R})$ .



- Multiresolution approximations

- The partial sum of wavelet coefficients giving  $d_j(t)$  can be interpreted as the difference between two approximations of  $f$  at the scales  $2^j$  and  $2^{(j-1)}$
- Multiresolution approximations compute the approximations of signals at various resolutions with orthogonal projections to different spaces  $\{V_j\}_{j \in \mathbb{Z}}$
- The **approximation of  $f$  at scale  $2^j$**  is specified by a discrete grid of samples that provides *local averages* of  $f$  on neighborhoods of size proportional to  $2^j$ .
- ***A multiresolution consists of embedded grids of approximations***

## Orthogonal wavelet bases

- The search for orthogonal wavelets begins with multiresolution approximations

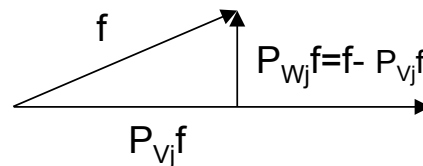
$$f \in L^2(\mathfrak{R}) \rightarrow \sum_{n=-\infty}^{+\infty} \langle f, \psi_{j,n} \rangle \psi_{j,n}$$

difference between two approximations  
at resolutions  $2^{-j+1}$  and  $2^{-j}$

- Resolution = 1/scale
  - The larger the scale, the smaller the resolution
- Multiresolution approximations compute the approximation of signals at various resolutions with orthogonal projections on different spaces  $\{V_j\}_{j \in \mathbb{Z}}$ 
  - These are characterized by a one particular discrete filter that governs the loss of information across resolutions

## Multiresolution approximations

- The approximation of a function  $f$  at a resolution  $2^j$  is specified by a discrete grid of samples that provides local averages of  $f$  over neighborhoods of size proportional to  $2^j$ .
- Thus, a multiresolution approximation is composed of *embedded grids of approximation*.
- More formally:
  - the approximation of a function at a resolution  $2^j$  is defined as an **orthogonal projection** on a space  $V_j \subset L^2(\mathbb{R})$ .
  - The space  $V_j$  regroups **all possible** approximations at the resolution  $2^j$ .
  - The orthogonal projection of  $f$  is the function  $f_j \in V_j$  that minimizes  $\|f - f_j\|$ .



# Multiresolution approximations

*Definition 7.1* A sequence  $\{V_j\}_{j \in \mathbb{Z}}$  of closed subspaces of  $L^2(\mathbb{R})$  is a multiresolution approximation if the following six conditions are satisfied

$$\forall (j, k) \in \mathbb{Z}^2, f(t) \in V_j \Leftrightarrow f(t - 2^j k) \in V_j$$

$V_j$  is invariant for translations proportional to the scale

$$\forall j \in \mathbb{Z}, V_{j+1} \subset V_j$$

The *finer* approximation subspace encloses all the information concerning the coarser one

$$\forall j \in \mathbb{Z}, f(t) \in V_j \Leftrightarrow f\left(\frac{t}{2}\right) \in V_{j+1}$$

Stretching the function by a factor 2 spans a coarser subspace

$$\lim_{j \rightarrow +\infty} V_j = \bigcap_{j=-\infty}^{+\infty} V_j = \{0\}$$


When the resolution goes to zero all the details are lost

$$\lim_{j \rightarrow +\infty} \|P_{V_j} f\| = 0.$$

$$\lim_{j \rightarrow -\infty} V_j = \text{Closure}\left(\bigcup_{j=-\infty}^{+\infty} V_j\right) = L^2(\mathbb{R})$$

When the resolution goes to infinity the approximation converges to the signal

$$\lim_{j \rightarrow -\infty} \|f - P_{V_j} f\| = 0.$$

There exists  $\vartheta$  such that  $\{\vartheta(t-n)\}_{n \in \mathbb{Z}}$  is a Riesz basis of  $V_0$  

discretization theorem

$j \leftrightarrow \text{scale}$   
 $2^{-j} \leftrightarrow \text{resolution}$

## Banach and Hilbert spaces

- A Hilbert space is an abstract vector space possessing the structure of an inner product that allows *length* and *angle* to be measured.
- Hilbert spaces are in addition required to be *complete*, a property that stipulates the existence of enough limits in the space to allow the techniques of calculus to be used.

## Banach and Hilbert spaces

- Banach space

Signals are often considered as vectors. To define a distance, we work within a vector space  $\mathbf{H}$  that admits a norm. A norm satisfies the following properties:

$$\forall f \in \mathbf{H}, \quad \|f\| \geq 0 \quad \text{and} \quad \|f\| = 0 \Leftrightarrow f = 0, \quad (\text{A.3})$$

$$\forall \lambda \in \mathbb{C} \quad \|\lambda f\| = |\lambda| \|f\|, \quad (\text{A.4})$$

$$\forall f, g \in \mathbf{H}, \quad \|f + g\| \leq \|f\| + \|g\|. \quad (\text{A.5})$$

With such a norm, the convergence of  $\{f_n\}_{n \in \mathbb{N}}$  to  $f$  in  $\mathbf{H}$  means that

$$\lim_{n \rightarrow +\infty} f_n = f \Leftrightarrow \lim_{n \rightarrow +\infty} \|f_n - f\| = 0.$$

To guarantee that we remain in  $\mathbf{H}$  when taking such limits, we impose a completeness property, using the notion of *Cauchy sequences*. A sequence  $\{f_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence if for any  $\varepsilon > 0$ , if  $n$  and  $p$  are large enough, then  $\|f_n - f_p\| < \varepsilon$ . The space  $\mathbf{H}$  is said to be *complete* if every Cauchy sequence in  $\mathbf{H}$  converges to an element of  $\mathbf{H}$ .

## Example 1

- For any integer  $p > 0$  we define over discrete sequences  $f[n]$

$$\|f\|_p = \left( \sum_{n=-\infty}^{+\infty} |f[n]|^p \right)^{1/p}$$

- The space

$$l^p = \left\{ f : \|f\|_p < +\infty \right\}$$

is a Banach space with the norm  $\|f\|_p$



## Example 2

**Example A.2** The space  $L^p(\mathbb{R})$  is composed of the measurable functions  $f$  on  $\mathbb{R}$  for which

$$\|f\|_p = \left( \int_{-\infty}^{+\infty} |f(t)|^p dt \right)^{1/p} < +\infty.$$

This integral defines a norm and  $L^p(\mathbb{R})$  is a Banach space, provided one identifies functions that are equal almost everywhere.

# Banach and Hilbert spaces

- Hilbert space

Whenever possible, we work in a space that has an inner product to define angles and orthogonality. A *Hilbert space*  $\mathbf{H}$  is a Banach space with an inner product. The inner product of two vectors  $\langle f, g \rangle$  is linear with respect to its first argument:

$$\forall \lambda_1, \lambda_2 \in \mathbb{C}, \quad \langle \lambda_1 f_1 + \lambda_2 f_2, g \rangle = \lambda_1 \langle f_1, g \rangle + \lambda_2 \langle f_2, g \rangle. \quad (\text{A.6})$$

It has an Hermitian symmetry:

$$\langle f, g \rangle = \langle g, f \rangle^*.$$

Moreover,

$$\langle f, f \rangle \geq 0 \quad \text{and} \quad \langle f, f \rangle = 0 \Leftrightarrow f = 0.$$

One can verify that  $\|f\| = \langle f, f \rangle^{1/2}$  is a norm. The positivity (A.3) implies the Cauchy-Schwarz inequality:

$$|\langle f, g \rangle| \leq \|f\| \|g\|, \quad (\text{A.7})$$

which is an equality if and only if  $f$  and  $g$  are linearly dependent.

We write  $\mathbf{V}^\perp$  the orthogonal complement of a subspace  $\mathbf{V}$  of  $\mathbf{H}$ . All vectors of  $\mathbf{V}^\perp$  are orthogonal to all vectors of  $\mathbf{V}$  and  $\mathbf{V} \oplus \mathbf{V}^\perp = \mathbf{H}$ .

## Example 3

**Example A.3** An inner product between discrete signals  $f[n]$  and  $g[n]$  can be defined by

$$\langle f, g \rangle = \sum_{n=-\infty}^{+\infty} f[n] g^*[n].$$

It corresponds to an  $\mathbf{l}^2(\mathbb{Z})$  norm:

$$\|f\|^2 = \langle f, f \rangle = \sum_{n=-\infty}^{+\infty} |f[n]|^2.$$

The space  $\mathbf{l}^2(\mathbb{Z})$  of finite energy sequences is therefore a Hilbert space. The Cauchy-Schwarz inequality (A.7) proves that

$$\left| \sum_{n=-\infty}^{+\infty} f[n] g^*[n] \right| \leq \left( \sum_{n=-\infty}^{+\infty} |f[n]|^2 \right)^{1/2} \left( \sum_{n=-\infty}^{+\infty} |g[n]|^2 \right)^{1/2}.$$

## Example 4

**Example A.4** Over analog signals  $f(t)$  and  $g(t)$ , an inner product can be defined by

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} f(t) g^*(t) dt.$$

The resulting norm is

$$\|f\| = \left( \int_{-\infty}^{+\infty} |f(t)|^2 dt \right)^{1/2}.$$

The space  $L^2(\mathbb{R})$  of finite energy functions is thus also a Hilbert space. In  $L^2(\mathbb{R})$ , the Cauchy-Schwarz inequality (A.7) is

$$\left| \int_{-\infty}^{+\infty} f(t) g^*(t) dt \right| \leq \left( \int_{-\infty}^{+\infty} |f(t)|^2 dt \right)^{1/2} \left( \int_{-\infty}^{+\infty} |g(t)|^2 dt \right)^{1/2}.$$

Two functions  $f_1$  and  $f_2$  are equal in  $L^2(\mathbb{R})$  if

$$\|f_1 - f_2\|^2 = \int_{-\infty}^{+\infty} |f_1(t) - f_2(t)|^2 dt = 0,$$

which means that  $f_1(t) = f_2(t)$  for almost all  $t \in \mathbb{R}$ .

# Bases of Hilbert spaces

## ***Orthonormal Basis***

A family  $\{e_n\}_{n \in \mathbb{N}}$  of a Hilbert space  $\mathbf{H}$  is orthogonal if for  $n \neq p$ ,

$$\langle e_n, e_p \rangle = 0.$$

If for  $f \in \mathbf{H}$  there exists a sequence  $a[n]$  such that

$$\lim_{N \rightarrow +\infty} \left\| f - \sum_{n=0}^N a[n] e_n \right\| = 0,$$

then  $\{e_n\}_{n \in \mathbb{N}}$  is said to be an *orthogonal basis* of  $\mathbf{H}$ . The orthogonality implies that necessarily  $a[n] = \langle f, e_n \rangle / \|e_n\|^2$ , and we write

$$f = \sum_{n=0}^{+\infty} \frac{\langle f, e_n \rangle}{\|e_n\|^2} e_n. \quad (\text{A.8})$$

A Hilbert space that admits an orthogonal basis is said to be *separable*.

The basis is *orthonormal* if  $\|e_n\| = 1$  for all  $n \in \mathbb{N}$ . Computing the inner product of  $g \in \mathbf{H}$  with each side of (A.8) yields a Parseval equation for orthonormal bases:

$$\langle f, g \rangle = \langle g, f \rangle^* \quad \langle f, g \rangle = \sum_{n=0}^{+\infty} \langle f, e_n \rangle \langle g, e_n \rangle^*. \quad (\text{A.9})$$

## Bases of Hilbert space

When  $g = f$ , we get an energy conservation called the *Plancherel formula*:

$$\|f\|^2 = \sum_{n=0}^{+\infty} |\langle f, e_n \rangle|^2. \quad (\mathbf{A.10})$$

The Hilbert spaces  $\ell^2(\mathbb{Z})$  and  $\mathbf{L}^2(\mathbb{R})$  are separable. For example, the family of translated Diracs  $\{e_n[k] = \delta[k - n]\}_{n \in \mathbb{Z}}$  is an orthonormal basis of  $\ell^2(\mathbb{Z})$ . Chapters 7 and 8 construct orthonormal bases of  $\mathbf{L}^2(\mathbb{R})$  with wavelets, wavelet packets, and local cosine functions.

# Multiresolution approximations

*Definition 7.1* A sequence  $\{V_j\}_{j \in \mathbb{Z}}$  of closed subspaces of  $L^2(\mathbb{R})$  is a multiresolution approximation if the following six conditions are satisfied

$$\forall (j, k) \in \mathbb{Z}^2, f(t) \in V_j \Leftrightarrow f(t - 2^j k) \in V_j$$

$V_j$  is invariant for translations proportional to the scale

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
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There exists  $\vartheta$  such that  $\{\vartheta(t-n)\}_{n \in \mathbb{Z}}$  is a Riesz basis of  $V_0$  

discretization theorem

$j \leftrightarrow \text{scale}$   
 $2^{-j} \leftrightarrow \text{resolution}$

# Riesz basis

Link to the discrete domain: the existence of a Riesz basis provides a *discretization theorem*

*Definition: A family of vectors is a Riesz basis of a space H if*

1. *it is linearly independent*
2. *there exist  $A, B > 0$  such that*

$$\forall y \in H \quad \exists \lambda[n]: \quad y = \sum_{n=0}^{+\infty} \lambda[n] e_n$$

$$\frac{1}{B} \|y\|^2 \leq \sum_{n=0}^{+\infty} |\lambda[n]|^2 \leq \frac{1}{A} \|y\|^2$$

*The existence of a Riesz basis for  $V_0$  provides a **discretization theorem**. There exists  $A$  and  $B$*

*such that any  $f \in V_0$  can be*

$$\forall f(t) \in V_0 \rightarrow f(t) = \sum_n a[n] \vartheta(t-n) \quad (7.9)$$

*uniquely decomposed into*

$$A \|f\|^2 \leq \sum_n |a[n]|^2 \leq B \|f\|^2 \quad (7.10)$$

$$(7.4) \quad \forall j \in \mathbb{Z}, f(t) \in V_j \Leftrightarrow f\left(\frac{t}{2}\right) \in V_{j+1} \rightarrow \left\{ \frac{1}{\sqrt{2^j}} \vartheta\left(\frac{t-2^j n}{2^j}\right) \right\}_{n \in \mathbb{Z}} \text{ is a Riesz basis for } V_j$$



## Riesz basis

- **Proposition 7.1** A family  $\{\vartheta(t-n)\}_{n \in \mathbb{Z}}$  is a Riesz basis of the space  $V_0$  it generates if and only if there are  $A > 0$  and  $B > 0$  such that

$$(7.11) \quad \forall \omega \in [-\pi, \pi], \frac{1}{B} \leq \sum_{k=-\infty}^{+\infty} |\vartheta(\omega - 2k\pi)|^2 \leq \frac{1}{A}$$

- Proof

- $\forall f \in V_0 \rightarrow f(t) = \sum_{n=-\infty}^{+\infty} a[n] \vartheta(t-n)$  taking the FT of both sides (7.12)

$$\hat{f}(\omega) = \hat{a}(\omega) \hat{\vartheta}(\omega)$$

Since  $a[n]$  is a Fourier series

$$\hat{a}(\omega) = \sum_{n=-\infty}^{+\infty} a[n] e^{-j\omega n} \quad \text{and is } 2\pi \text{ periodic, hence}$$

$$\hat{a}(\omega) = \sum_{n=-\infty}^{+\infty} \hat{a}(\omega - n2\pi) = \hat{a}(\omega) * \sum_{n=-\infty}^{+\infty} \delta(\omega - n2\pi)$$

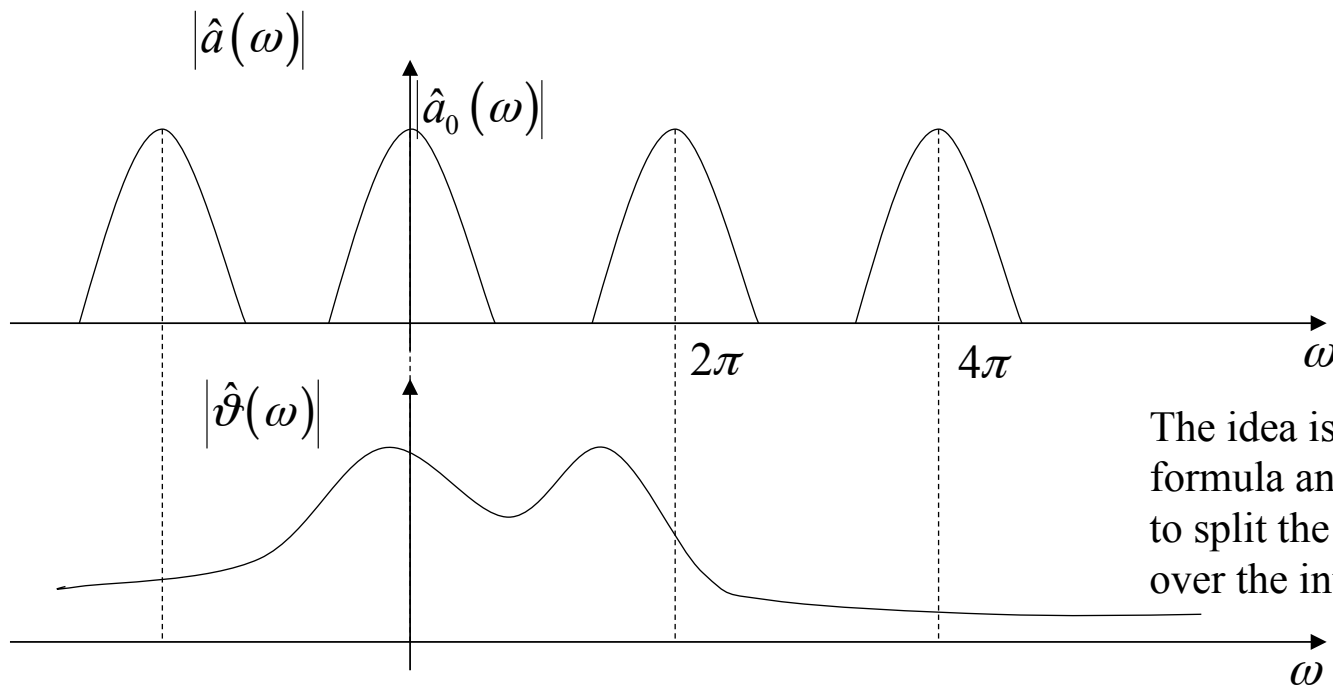
## Intuition (1)

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{f}(\omega)|^2 d\omega.$$

$$\hat{a}(\omega) = \sum_{n=-\infty}^{+\infty} \hat{a}(\omega - n2\pi) = \hat{a}(\omega) * \sum_{n=-\infty}^{+\infty} \delta(\omega - n2\pi)$$

- Applying the definition of norm (Plancherel)

$$\|\hat{f}(\omega)\|^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{f}(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{a}(\omega)|^2 |\hat{\vartheta}(\omega)|^2 d\omega \quad (1)$$



The idea is to exploit the Plancherel's formula and the fact that  $a(\omega)$  is periodic to split the integral into sums of integrals over the interval  $0-2\pi$ .

## Proof (1)

- Using Plancherel formula and the fact that  $a(\omega)$  is periodic (see Mallat version 2009 page 67)

$$\|\hat{f}(\omega)\|^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{f}(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{a}(\omega)|^2 |\hat{\vartheta}(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{a}(\omega) \hat{\vartheta}(\omega)|^2 d\omega =$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \sum_k \hat{a}(\omega - 2k\pi) \hat{\vartheta}(\omega) \right|^2 d\omega =$$
$$= \frac{1}{2\pi} \int_0^{2\pi} |\hat{a}(\omega)|^2 \sum_{k=-\infty}^{+\infty} |\hat{\vartheta}(\omega - 2k\pi)|^2 d\omega$$

since  $a(\omega)$  is periodic, taking the integral over subsequent intervals amounts only to “shifting” the second function. The first,  $a(\omega)$ , remains the same so it can be taken out of the sum.

## Proof (2)

- Norm

$$\|\hat{f}(\omega)\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |\hat{a}(\omega)|^2 \sum_{k=-\infty}^{+\infty} |\hat{\vartheta}(\omega - 2k\pi)|^2 d\omega$$

$$\forall \omega \in [-\pi, \pi] \quad \frac{1}{B} \leq \sum_{k=-\infty}^{+\infty} |\vartheta(\omega - 2k\pi)|^2 \leq \frac{1}{A} \quad \text{then} \quad (2)$$

$$\|f(t)\|^2 \leq \frac{1}{A} \frac{1}{2\pi} \int_0^{2\pi} |a(\omega)|^2 d\omega = \frac{1}{A} \sum_{n=-\infty}^{+\infty} |a[n]|^2 \rightarrow$$

$$A \|f(t)\|^2 \leq \sum_{n=-\infty}^{+\infty} |a[n]|^2$$

To note: (1) is a function of omega thus the condition (2) means that the pointwise sum of the values of the translates of the function in omega is finite.

$$\sum_{k=-\infty}^{+\infty} |\vartheta(\omega - 2k\pi)|^2 \quad (1)$$

For A=B=1 the basis is orthonormal and (2) takes the definition of “partition of unity”. This is the case for the scaling function.

## Proof (3)

- Similarly

$$B\|f(t)\|^2 \geq \sum_{n=-\infty}^{+\infty} |a[n]|^2$$

- Thus

$$(7.15) \quad A\|f(t)\|^2 \leq \sum_{n=-\infty}^{+\infty} |a[n]|^2 \leq B\|f(t)\|^2$$

In summary, if  $\theta(t-n)$  satisfies (7.11 Mallat 99) then (7.15) is satisfied. Then,  $\theta(t-n)$  is a Riesz basis for  $V_0$  and every function in  $V_0$  can be expressed as in (7.12)

$$f(t) = \sum_{k=-\infty}^{+\infty} a[k]\vartheta(t-n) \quad (7.12)$$

# Scaling function

- The **scaling function** is obtained by the **orthogonalization of the Riesz basis**

Theorem 7.1

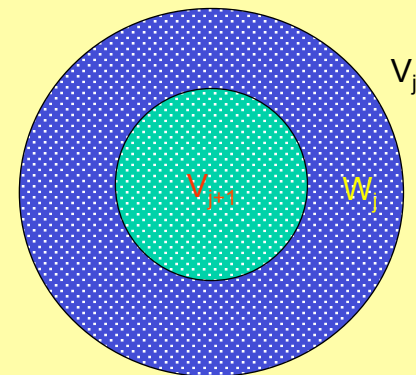
Let  $V_j$  be a multiresolution approximation and  $\varphi$  be the scaling function whose FT is

$$\hat{\varphi}(\omega) = \frac{\hat{\vartheta}(\omega)}{\left( \sum_{k=-\infty}^{+\infty} |\hat{\vartheta}(\omega + 2k\pi)|^2 \right)^{1/2}}$$

Let us denote

$$\varphi_{j,n}(t) = \frac{1}{\sqrt{2^j}} \varphi\left(\frac{t - 2^j n}{2^j}\right)$$

The family  $\{\varphi_{j,n}\}_{n \in \mathbb{Z}}$  is an orthonormal basis of  $V_j$  for all  $j$  in  $\mathbb{Z}$



# Proof

*Proof*<sup>1</sup>. To construct an orthonormal basis, we look for a function  $\phi \in \mathbf{V}_0$ . It can thus be expanded in the basis  $\{\theta(t-n)\}_{n \in \mathbf{Z}}$ :

$$\phi(t) = \sum_{n=-\infty}^{+\infty} a[n] \theta(t-n),$$

which implies that

$$\hat{\phi}(\omega) = \hat{a}(\omega) \hat{\theta}(\omega),$$

where  $\hat{a}$  is a  $2\pi$  periodic Fourier series of finite energy. To compute  $\hat{a}$  we express the orthogonality of  $\{\phi(t-n)\}_{n \in \mathbf{Z}}$  in the Fourier domain. Let  $\bar{\phi}(t) = \phi^*(-t)$ . For any  $(n, p) \in \mathbf{Z}^2$ ,

$$\begin{aligned} \langle \phi(t-n), \phi(t-p) \rangle &= \int_{-\infty}^{+\infty} \phi(t-n) \phi^*(t-p) dt \\ &= \phi \star \bar{\phi}(p-n). \end{aligned} \quad (1) \quad (7.18)$$

Hence  $\{\phi(t-n)\}_{n \in \mathbf{Z}}$  is orthonormal if and only if  $\phi \star \bar{\phi}(n) = \delta[n]$ . Computing the Fourier transform of this equality yields

$$\sum_{k=-\infty}^{+\infty} |\hat{\phi}(\omega + 2k\pi)|^2 = 1. \quad (7.19)$$

Indeed, the Fourier transform of  $\phi \star \bar{\phi}(t)$  is  $|\hat{\phi}(\omega)|^2$ , and we proved in (3.3) that sampling a function periodizes its Fourier transform. The property (7.19) is verified if we choose

$$\hat{a}(\omega) = \left( \sum_{k=-\infty}^{+\infty} |\hat{\theta}(\omega + 2k\pi)|^2 \right)^{-1/2}.$$

Proposition 7.1 proves that the denominator has a strictly positive lower bound, so  $\hat{a}$  is a  $2\pi$  periodic function of finite energy. ■

Thus here we apply the same idea as in the previous proof: relying on Plancherel formula and explicating the fact that the function is periodic in the Fourier domain. Thus, replacing the result in (1) we get the orthogonalization formula.

# Approximation

- The orthogonal projection of  $f$  onto  $V_j$  is obtained as an expansion in the scaling orthogonal basis

$$P_{V_j} f = \sum_{n=-\infty}^{+\infty} \langle f, \varphi_{j,n} \rangle \varphi_{j,n}$$

- The inner products  $a_j[n]$  are the projection coefficients at scale  $2^j$

$$a_j[n] = \langle f, \varphi_{j,n} \rangle = \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{2^j}} \varphi\left(\frac{t-2^j n}{2^j}\right) dt = f * \bar{\varphi}_j(2^j n)$$

$$\bar{\varphi}_j(t) = \frac{1}{\sqrt{2^j}} \varphi\left(-\frac{t}{2^j}\right)$$

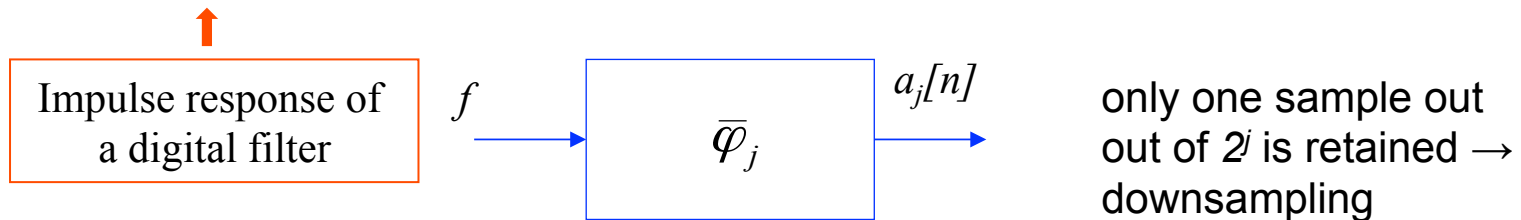
- As proved in what above, the normalization factor at the denominator ensures that

$$\hat{\varphi}(\omega) = \frac{\hat{\vartheta}(\omega)}{\left(\sum_{k=-\infty}^{+\infty} |\hat{\vartheta}(\omega + 2k\pi)|^2\right)^{1/2}} \quad \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\omega + 2k\pi)|^2 = 1 \quad \textit{partition of unity}$$



# Approximation

$$a_j[n] = f * \bar{\varphi}_j(2^j n)$$



- The energy of  $\varphi_j$  is mostly concentrated in  $[-\pi/2^j, \pi/2^j]$  which corresponds to low pass filtering
- The *signal approximation* is obtained by convolving  $f$  with a *low-pass filter* and downsampling by 2 -> any scaling function corresponds to a *conjugate mirror filter*
- A multiresolution is *completely characterized* by the scaling function

# Wavelet representation

- Summarizing

$$A^d_{2^j} f = PV_j f = \sum_{n=-\infty}^{+\infty} \langle f, \varphi_{j,n} \rangle \varphi_{j,n}$$

discrete approximation at resolution j

$$a_j[n] = \langle f, \varphi_{j,n} \rangle$$

discrete approximation coefficients at resolution j

$$d_{2^j} f = PW_j f = \sum_{n=-\infty}^{+\infty} \langle f, \psi_{j,n} \rangle \psi_{j,n}$$

details at resolution j

$$d_j[n] = \langle f, \psi_{j,n} \rangle$$

wavelet coefficients at resolution j

$$\left\{ A^d_{2^J} f, \left\{ d_{2^j} f \right\}_{1 \leq j \leq J} \right\}$$

wavelet representation

## Analytic versus real wavelets

- Real wavelets are used to detect sharp signal transitions
- Analytic wavelets can measure the time evolution of a frequency gradient, as they allow to separate the phase and amplitude information
- Fourier analogy
  - DCT: real basis functions
  - DFT: complex basis functions

DCT describes all (symmetrized) signals as a linear combination of cosinusoids such that the phase information is lost.

On the contrary, complex exponentials preserve the information about the phase

## Analytic signals

- A function  $f_a(x) \in L^2(\mathbb{R})$  is said to be analytic if its Fourier transform is zero for negative frequencies

$$\hat{f}_a(\omega) = 0 \quad \text{if } \omega < 0$$

- An analytic function is necessarily complex but it is completely characterized by its real part  $\text{Re}[f_a(\omega)]$

$$\hat{f}(\omega) = \frac{\hat{f}_a(\omega) + \hat{f}_a^*(-\omega)}{2} \Leftrightarrow \hat{f}_a(\omega) = \begin{cases} 2\hat{f}(\omega) & \text{if } \omega \geq 0 \\ 0 & \text{if } \omega < 0 \end{cases}$$

## Discrete analytic part

**Discrete Analytic Part** The analytic part  $f_a[n]$  of a discrete signal  $f[n]$  of size  $N$  is also computed by setting to zero the negative frequency components of its discrete Fourier transform. The Fourier transform values at  $k = 0$  and  $k = N/2$  must be carefully adjusted so that  $\text{Real}[f_a] = f$ :

$$\hat{f}_a[k] = \begin{cases} \hat{f}[k] & \text{if } k = 0, N/2 \\ 2\hat{f}[k] & \text{if } 0 < k < N/2 \\ 0 & \text{if } N/2 < k < N \end{cases} . \quad (4.48)$$

We obtain  $f_a[n]$  by computing the inverse discrete Fourier transform.

## Example

**Example 4.8** The Fourier transform of

Real function

$$\Rightarrow f(t) = a \cos(\omega_0 t + \phi) = \frac{a}{2} \left( \exp[i(\omega_0 t + \phi)] + \exp[-i(\omega_0 t + \phi)] \right)$$

is

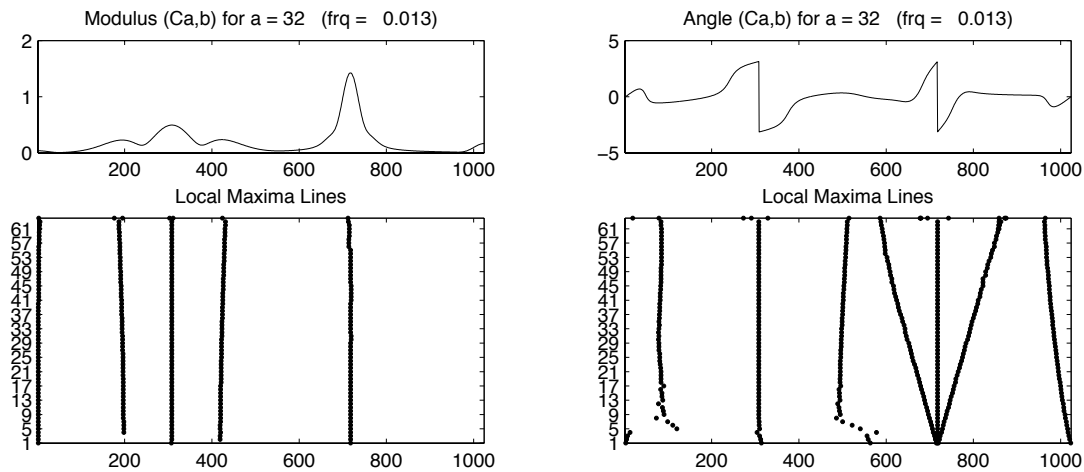
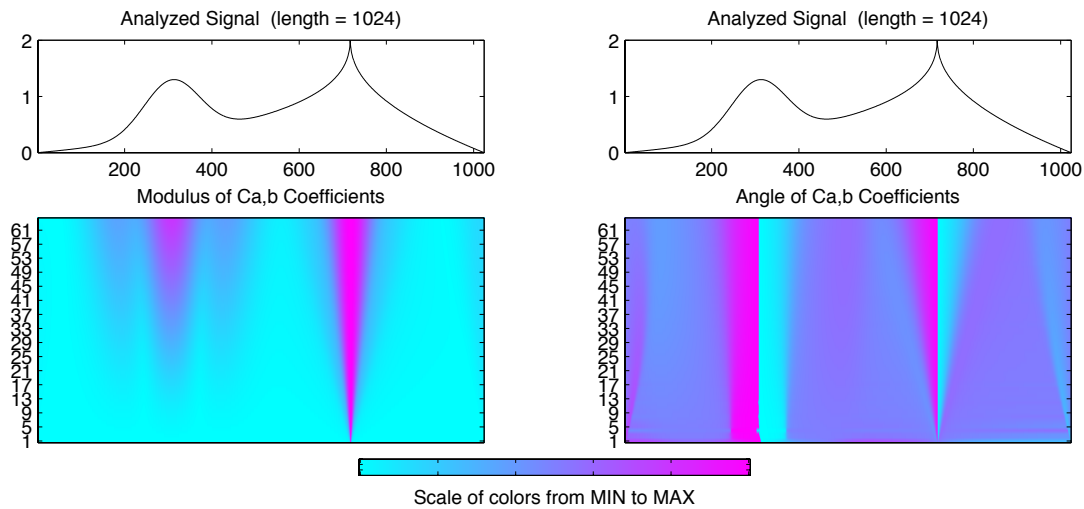
$$\hat{f}(\omega) = \pi a \left( \exp(i\phi) \delta(\omega - \omega_0) + \exp(-i\phi) \delta(\omega + \omega_0) \right).$$

The Fourier transform of the analytic part computed with (4.47) is  $\hat{f}_a(\omega) = 2\pi a \exp(i\phi) \delta(\omega - \omega_0)$  and hence

$$\Rightarrow f_a(t) = a \exp[i(\omega_0 t + \phi)]. \quad (4.49)$$

Complex function

# Example: analytic wavelet



Data (Size)

Wavelet

Sampling Period

Scale Settings

Step by Step Mode

Min (> 0)

Step (> 0)

Max (<= 512)

Analyze

New Coefficients Line

Refresh Maxima Lines

Selected Axes

Modulus  Angle  Both

Coefficients

Coefficients Line

Maxima Lines

Scales  Frequencies

Coloration Mode

init + by scale

Colormap

Nb. Colors

Brightness

Close

X+ Y+ XY+ Center On X Y Info X = Y = History < > << >> View Axes

X- Y- XY-

# Wavelets and multiresolution representations



## Scaling equation

- A multiresolution approximation is completely characterized by the function  $\varphi$  that generates the orthonormal bases for each  $V_j$
- We study the properties of  $\varphi$  which guarantee that all the spaces  $V_j$  satisfy all conditions of a multiresolution approximation.
- It is proved that **any scaling function corresponds to a discrete filter called conjugate mirror filter**
- Procedure
  1. Link  $\varphi$  to the corresponding discrete filter  $h[n]$
  2. Determine the properties of  $h[n]$  such that  $\varphi$  is a scaling function

## Scaling equation

- From multiresolution conditions follows

$$V_j \subset V_{j-1}$$

$$\frac{1}{\sqrt{2}} \varphi\left(\frac{t}{2}\right) \in V_1 \subset V_0$$

$$\frac{1}{\sqrt{2}} \varphi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} h[n] \varphi(t-n) \quad (1)$$

$$h[n] = \left\langle \frac{1}{\sqrt{2}} \varphi\left(\frac{t}{2}\right), \varphi(t-n) \right\rangle$$

$f(t)$

- The **scaling equation** relates a dilation of  $\varphi$  by 2 to its integer translations.
- The sequence  $h[n]$  will be interpreted as a discrete filter

## Scaling equation

- Taking the F-trasform of (1)

$$\mathfrak{F}\left\{\frac{1}{\sqrt{2}}\varphi\left(\frac{t}{2}\right)\right\} = \mathfrak{F}\left\{\sum_{n=-\infty}^{+\infty} h[n]\varphi(t-n)\right\} \rightarrow$$

convolution product

$$\hat{\phi}(2\omega) = \frac{1}{\sqrt{2}}\hat{h}(\omega)\hat{\phi}(\omega) \quad (2)$$

- where

$$\hat{h}(\omega) = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n}$$

- Next step is thus the expression of  $\hat{\varphi}(\omega)$  as a product of dilations of  $\hat{h}(\omega)$ .
  - For any  $p \geq 0$ , (2) implies

$$\hat{\phi}\left(2^{-p+1}\omega\right) = \frac{1}{\sqrt{2}}\hat{h}\left(2^{-p}\omega\right)\hat{\phi}\left(2^{-p}\omega\right) \iff \hat{\phi}\left(\frac{\omega}{2^{p-1}}\right) = \frac{1}{\sqrt{2}}\hat{h}\left(\frac{\omega}{2^p}\right)\hat{\phi}\left(\frac{\omega}{2^p}\right)$$

## Scaling equation

Iterating (2):

$$\hat{\Phi}(2\omega) = \frac{1}{\sqrt{2}} \hat{h}(\omega) \hat{\Phi}(\omega) \rightarrow$$

$$\hat{\Phi}(\omega) = \frac{1}{\sqrt{2}} \hat{h}\left(\frac{\omega}{2}\right) \hat{\Phi}\left(\frac{\omega}{2}\right), \rightarrow \hat{\Phi}\left(\frac{\omega}{2}\right) = \frac{1}{\sqrt{2}} \hat{h}\left(\frac{\omega}{4}\right) \hat{\Phi}\left(\frac{\omega}{4}\right) \rightarrow \dots \hat{\Phi}(2^{-p+1}\omega) = \hat{h}(2^{-p}\omega) \hat{\Phi}(2^{-p}\omega)$$

replacing in the expression above for all values of p up to P:

$$\hat{\Phi}(\omega) = \left(\frac{1}{\sqrt{2}}\right)^2 \hat{\Phi}\left(\frac{\omega}{4}\right) \hat{h}\left(\frac{\omega}{4}\right) \hat{h}\left(\frac{\omega}{2}\right)$$

.....

$$\hat{\Phi}(\omega) = \prod_{p=1}^P \frac{\hat{h}(2^{-p}\omega)}{\sqrt{2}} \hat{\Phi}(2^{-P}\omega)$$

If  $\hat{\varphi}(\omega)$  is continuous at  $\omega=0$  then

$$\lim_{P \rightarrow +\infty} \left( \hat{\Phi}(2^{-P}\omega) \right) = \hat{\Phi}(0) \rightarrow$$

$$\hat{\Phi}(\omega) = \prod_{p=1}^{+\infty} \frac{\hat{h}(2^{-p}\omega)}{\sqrt{2}} \hat{\Phi}(0)$$

→ Next step: find the necessary and sufficient conditions on  $\hat{h}(\omega)$  to guarantee that this infinite product is the F-transform of a scaling function

# Conjugate Mirror Filters

## Teorem 7.2 (Mallat&Meyer)

Let  $\varphi$  in  $L^2(\mathbb{R})$  be an integrable scaling function. The F-series of  $h[n]$  satisfies

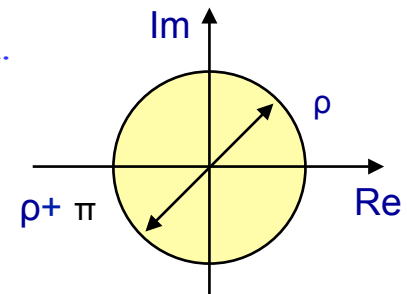
$$(2) \quad \forall \omega \quad \left| \hat{h}(\omega) \right|^2 + \left| \hat{h}(\omega + \pi) \right|^2 = 2 \quad \text{and} \quad \hat{h}(0) = \sqrt{2} \quad \text{CMF}$$

**Conversely**, if  $\hat{h}(\omega)$  is  $2\pi$  periodic and continuously differentiable in a neighborhood of  $\omega=0$ , if it satisfies (2) and if

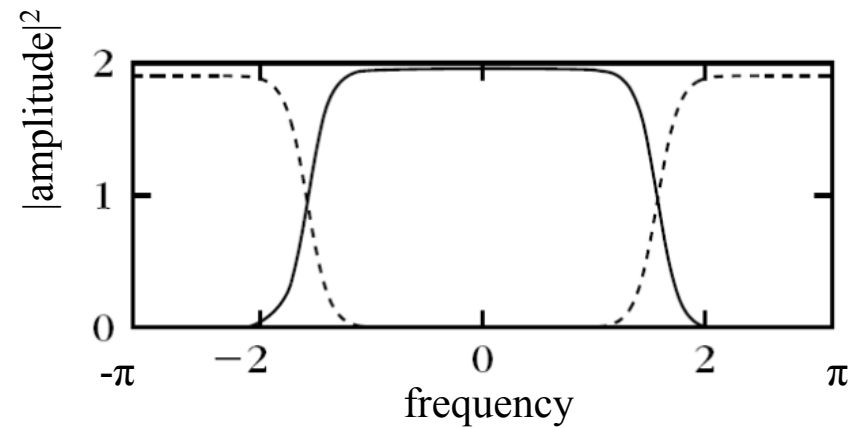
$$\inf_{\omega \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]} \left| \hat{h}(\omega) \right| > 0 \quad \text{It does not vanish at } \omega=0$$

Then,  $\hat{\Phi}(\omega) = \prod_{p=1}^{+\infty} \frac{\hat{h}(2^{-p}\omega)}{\sqrt{2}}$  is the F-transform of a scaling function.

This theorem provides the conditions under which the **discrete filter  $h[n]$  generates a scaling function** or, equivalently, a multiresolution representation frame.



## CMF property



The solid line gives  $|\hat{h}(\omega)|^2$  on  $[-\pi, \pi]$  for a cubic spline multiresolution. The dotted line corresponds to  $|\hat{g}(\omega)|^2$ , namely the corresponding band-pass filter.

## Conjugate mirror filters

**Table 7.1** Conjugate Mirror Filters  $h[n]$  for Linear Splines  $m = 1$  and Cubic Splines  $m = 3$

	$n$	$h[n]$		$n$	$h[n]$
$m = 1$	0	0.817645956	$m = 3$	5, -5	0.042068328
	1, -1	0.397296430		6, -6	-0.017176331
	2, -2	-0.069101020		7, -7	-0.017982291
	3, -3	-0.051945337		8, -8	0.008685294
	4, -4	0.016974805		9, -9	0.008201477
	5, -5	0.009990599		10, -10	-0.004353840
	6, -6	-0.003883261		11, -11	-0.003882426
	7, -7	-0.002201945		12, -12	0.002186714
	8, -8	0.000923371		13, -13	0.001882120
	9, -9	0.000511636		14, -14	-0.001103748
	10, -10	-0.000224296		15, -15	-0.000927187
11, -11	-0.000122686	16, -16	0.000559952		
$m = 3$	0	0.766130398	17, -17	0.000462093	
	1, -1	0.433923147	18, -18	-0.000285414	
	2, -2	-0.050201753	19, -19	-0.000232304	
	3, -3	-0.110036987	20, -20	0.000146098	
	4, -4	0.032080869			

*Note: The coefficients below  $10^{-4}$  are not given.*

## What about wavelets?

- Orthonormal wavelets carry the details needed to increase the resolution of a signal approximation.
- The approximations of  $f$  at scales  $2^j$  and  $2^{(j+1)}$  are respectively equal to its orthogonal projections in  $V_j$  and  $V_{j+1}$
- We know that  $V_{j+1}$  is included in  $V_j$

- Let  $W_{j+1}$  be the *orthogonal complement* of  $V_{j+1}$  in  $V_j$

$$V_{j-1} = V_j \oplus W_j$$

- The orthogonal projection of  $f$  on  $V_j$  can be decomposed as follows

$$PV_{j-1}f = PV_jf + PW_jf$$

- The complement  $PW_{j+1}f$  provides the details that appear at scale  $j$  but disappear at the next coarser scale.
- Next theorem will show that basis for  $W_j$  can be constructed by scaling and translating a wavelet  $\psi$



## Corresponding orthogonal wavelet family

- Theorem 7.3 [Mallat&Meyer]

Let  $\varphi$  be a scaling function and  $h$  the corresponding CMF. Let  $\Psi$  be such that

$$\hat{\Psi}(\omega) = \frac{1}{\sqrt{2}} \hat{g}\left(\frac{\omega}{2}\right) \hat{\Phi}\left(\frac{\omega}{2}\right)$$

with

$$\hat{g}(\omega) = e^{-j\omega} \hat{h}^*(\omega + \pi)$$

Let us denote

$$\psi_{j,n}(t) = \frac{1}{\sqrt{2^j}} \psi\left(\frac{t - 2^j n}{2^j}\right)$$

For any scale,  $\{\Psi_{j,n}\}_{j \in \mathbb{Z}}$  is an orthonormal basis for  $W_j$ .

For all  $j$ ,  $\{\psi_{j,n}\}_{j,n \in \mathbb{Z}^2}$  is an orthonormal basis for  $L^2$ .

Signal domain

$$\hat{g}(\omega) = e^{-j\omega} \hat{h}^*(\omega + \pi) \leftrightarrow g(z) = z^{-1} h(-z^{-1}) \leftrightarrow g[n] = (-1)^{1-n} h[1-n]$$

The frequency modulation changes the low-pass filter  $h[n]$  to the band-pass filter  $g[n]$   
The phase modulation introduces a unitary step delay

## Proof

$$h(\omega) \rightarrow h[n]$$

$$h(\omega + \pi) \rightarrow (-1)^n h[n]$$

$$h^*(\omega + \pi) = h(-(\omega + \pi)) \rightarrow (-1)^{-n} h[-n]$$

$$e^{-j\omega} h^*(\omega + \pi) \rightarrow (-1)^{1-n} h[1-n]$$

1. Frequency modulation changes each other sample sign
2. Phase reverse changes n in -n
3. Phase modulation introduces a unit delay

$$h(\omega) = \dots + h[0] + h[1]e^{-j\omega} + h[2]e^{-2j\omega} + h[3]e^{-3j\omega} + \dots$$

$$h(\omega + \pi) = \dots + h[0] + h[1]e^{-j(\omega+\pi)} + h[2]e^{-2j(\omega+\pi)} + h[3]e^{-3j(\omega+\pi)} + \dots =$$

$$= \dots + h[0] - h[1]e^{-j\omega} + h[2]e^{-2j\omega} - h[3]e^{-3j\omega} + \dots = \sum_n (-1)^n h[n] e^{-j\omega n}$$

$$h^*(\omega + \pi) = \sum_n (-1)^n h[n] e^{j\omega n} = \sum_n (-1)^{-n} h[-n] e^{-j\omega n} = \mathfrak{S}\{(-1)^{-n} h[-n]\}$$

$$e^{-j\omega} h^*(\omega + \pi) \rightarrow (-1)^{1-n} h[1-n]$$

## Corresponding orthogonal wavelet family

- Lemma 7.1. The family  $\{\psi_{j,n}\}_{n \in \mathbb{Z}}$  is an orthonormal basis for  $W_j$  iff

$$|\hat{g}(\omega)|^2 + |\hat{g}(\omega + \pi)|^2 = 2$$

and

$$\hat{g}(\omega)\hat{h}^*(\omega) + \hat{g}(\omega + \pi)\hat{h}^*(\omega + \pi) = 2$$

- Furthermore

$$V_{j-1} = V_j + W_j \rightarrow \frac{1}{\sqrt{2}}\psi\left(\frac{t}{2}\right) \in W_1 \subset V_0$$

since  $\{\varphi(t-n)\}_{n \in \mathbb{Z}}$  is an orthonormal basis of  $V_0 \rightarrow$

$$\frac{1}{\sqrt{2}}\psi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} g[n]\varphi(t-n) \quad \text{with}$$

$$g[n] = \left\langle \frac{1}{\sqrt{2}}\psi\left(\frac{t}{2}\right), \varphi(t-n) \right\rangle$$

- The orthogonal wavelets **carry the details lost going from scale  $j$  to scale  $j+1$**
- Wavelets are the **basis functions for  $W_j$**
- The details at scale  $j$  are obtained by **projecting the signal onto the wavelet family  $\psi_{j,n}$**

## Summary

- Approximation function at scale  $2^j$ :

$$P_{V_j} f = \sum_{n=-\infty}^{+\infty} \langle f, \varphi_{j,n} \rangle \varphi_{j,n}$$

- Details (“residual” functions) at scale  $2^j$ :

$$P_{W_j} f = \sum_{n=-\infty}^{+\infty} \langle f, \psi_{j,n} \rangle \psi_{j,n}$$

- Wavelet representation:

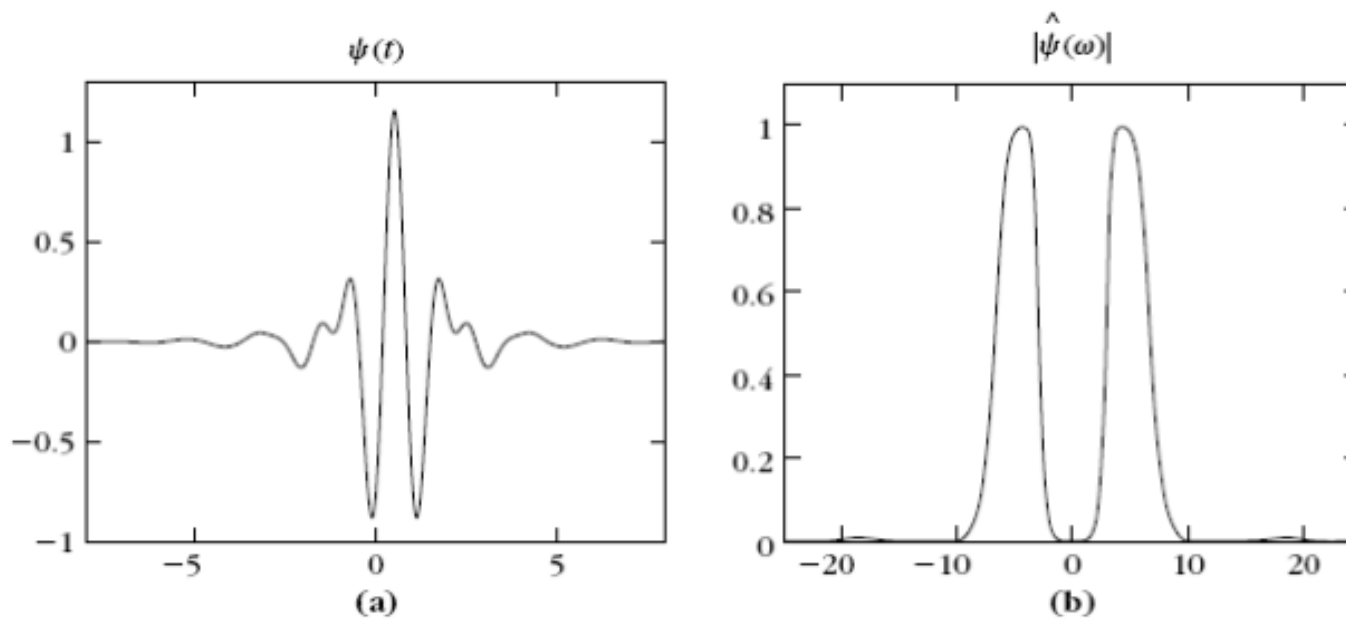
$$f = \sum_{j=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \langle f, \psi_{j,n} \rangle \psi_{j,n}$$

- If the basis is orthogonal, the scaling function characterizes the multi-resolution completely

Scaling function  $\varphi \rightarrow h[n] \rightarrow g[n] \rightarrow$  wavelet  $\psi$

## Example

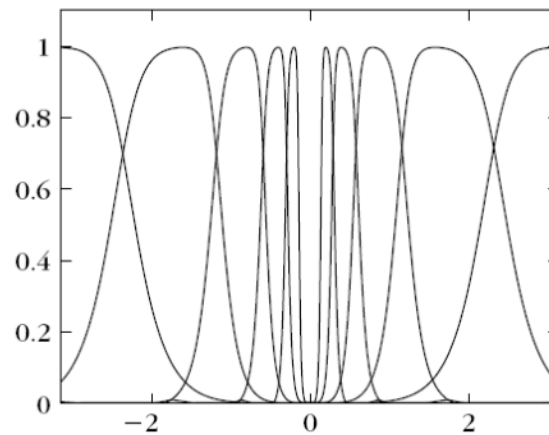
- Battle-Lemarié cubic spline wavelet and its spectrum



## Wavelet property

- Property: for any  $\psi$  that can generate an orthonormal family, one can verify that

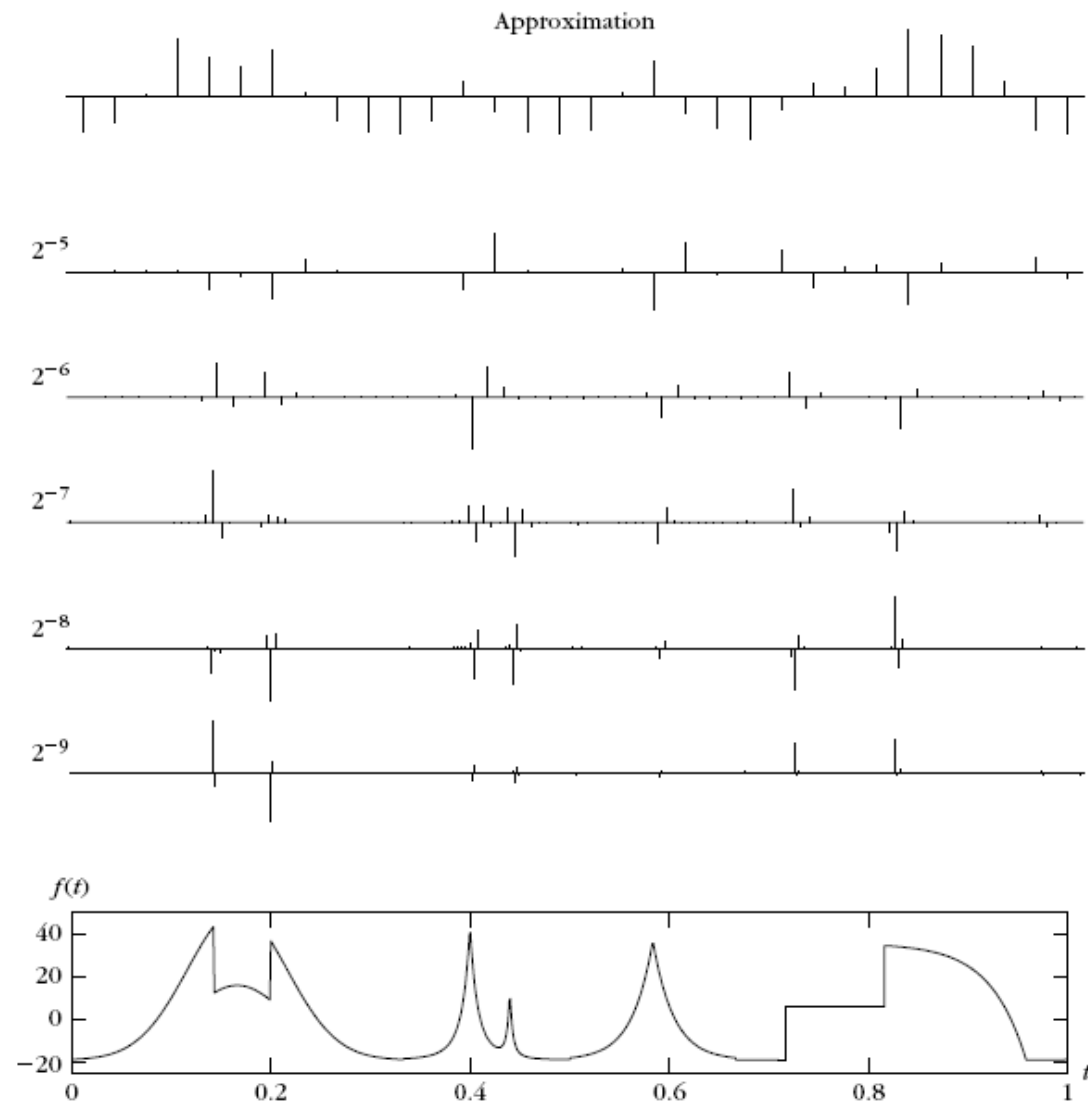
$$\forall \omega \in \mathbb{R} - \{0\}, \quad \sum_{j=-\infty}^{+\infty} |\hat{\psi}(2^j \omega)|^2 = 1$$



**FIGURE 7.6**

Graph of  $|\hat{\psi}(2^j \omega)|^2$  for the cubic spline Battle-Lemarié wavelet, with  $1 \leq j \leq 5$  and  $\omega \in [-\pi, \pi]$ .

# Example



**FIGURE 7.7**

Wavelet coefficients  $d_j[n] = \langle f, \psi_{j,n} \rangle$  calculated at scales  $2^j$  with the cubic spline wavelet. Each up or down Dirac gives the amplitude of a positive or negative wavelet coefficient. At the top is the remaining coarse-signal approximation  $a_J[n] = \langle f, \phi_{J,n} \rangle$  for  $J = -5$ .

## Warning

- Each CMF generates a wavelet orthonormal bases
- Does any **wavelet orthonormal bases** correspond to a multiresolution approximation and CMF? It depends on the support:
  - **If  $\psi$  has compact support than it corresponds to a multiresolution approximation** [Lemarié]
  - However, there exists “pathological” wavelets that decay as  $|t|^{-1}$  that cannot be derived from any multiresolution approximation



## Classes of wavelet bases

- Wavelets are interesting for applications for their ability to represent signals with **few non zero coefficients**
- The best basis for an application is the one that maximizes the number of zero or close to zero coefficients (sparsity). This depends on
  - The *regularity* of  $f$
  - The number of *vanishing moments* of the wavelet
  - The *size* of its support
- The constraints on the wavelet translate to **design rules for the filter  $g[n]$ , thus  $h[n]$** 
  - Thus, we need conditions on  $\hat{h}(\omega)$  based on wavelets properties

## Wavelet properties: vanishing moments

- Vanishing moments

- The wavelet has  $p$  vanishing moments if

$$\int_{-\infty}^{+\infty} t^k \psi(t) dt = 0 \quad \text{for } 0 \leq k < p \quad (3)$$

- The number of vanishing moments is equal to the *multiplicity of zeros of  $\hat{h}(\omega)$  in  $\pi$*  or, equivalently, the *number of vanishing derivatives of  $\hat{\psi}$  in zero*

- Theorem 7.4: Vanishing moments

*Let  $\varphi$  and  $\psi$  be a scaling function and a wavelet that generate an orthonormal basis. Suppose that  $|\psi(t)| = O((1+t^2)^{-p/2-1})$  and  $|\varphi(t)| = O((1+t^2)^{-p/2-1})$ . The four following statements are equivalent*

1. *The wavelet  $\psi$  has  $p$  vanishing moments*
2.  *$\hat{\psi}(\omega)$  and its first  $p-1$  derivatives are zero at  $\omega=0$*
3.  *$\hat{h}(\omega)$  and its first  $p-1$  derivatives are zero at  $\omega=\pi$*
4. *for any  $0 \leq k < p$   $q_k(t) = \sum_{n=-\infty}^{+\infty} n^k \varphi(t-n)$  is a polynomial of degree  $k$*

## hints of the proof

- Point 1. The decay of  $|\varphi(t)|$  and  $|\psi(t)|$  imply that  $|\hat{\varphi}(\omega)|$  and  $|\hat{\psi}(\omega)|$  are  $p$ -times differentiable
- Point 2. The  $k$ -th order derivative of  $\hat{\psi}^{(k)}(\omega)$  is the F-transform of  $(-it)^k \psi(t)$   
thus

$$\hat{\psi}^{(k)}(\omega) = \int_{-\infty}^{+\infty} (-it)^k \psi(t) dt. \quad (4)$$

(4) is equivalent to (3), which proves 2.

- Point 3.

$$\hat{\Psi}(\omega) = \frac{1}{\sqrt{2}} \hat{g}\left(\frac{\omega}{2}\right) \hat{\Phi}\left(\frac{\omega}{2}\right) \quad \hat{g}(\omega) = e^{-j\omega} \hat{h}^*(\omega + \pi) \quad \text{thus}$$

$$\hat{\Psi}(2\omega) = \frac{1}{\sqrt{2}} \hat{g}(\omega) \hat{\Phi}(\omega) = e^{-i\omega} \hat{h}^*(\omega + \pi) \hat{\Phi}(\omega)$$

since  $\hat{\Phi}(0) \neq 0$  by differentiating this expression we prove that 2. is equivalent to 3.

- Finally, it is proved that 4. is equivalent to 1. and viceversa.

## Strang-Fix conditions

- Let  $\varphi(t)$  satisfy the Riesz basis condition. Then, the following Strang-Fix conditions of order  $L$  are equivalent

$$\hat{\varphi}(0) = 1, \text{ and } \hat{\varphi}^{(n)}(2\pi k) = 0 \text{ for } \begin{cases} k \neq 0 \\ n = 0 \dots L-1 \end{cases}$$

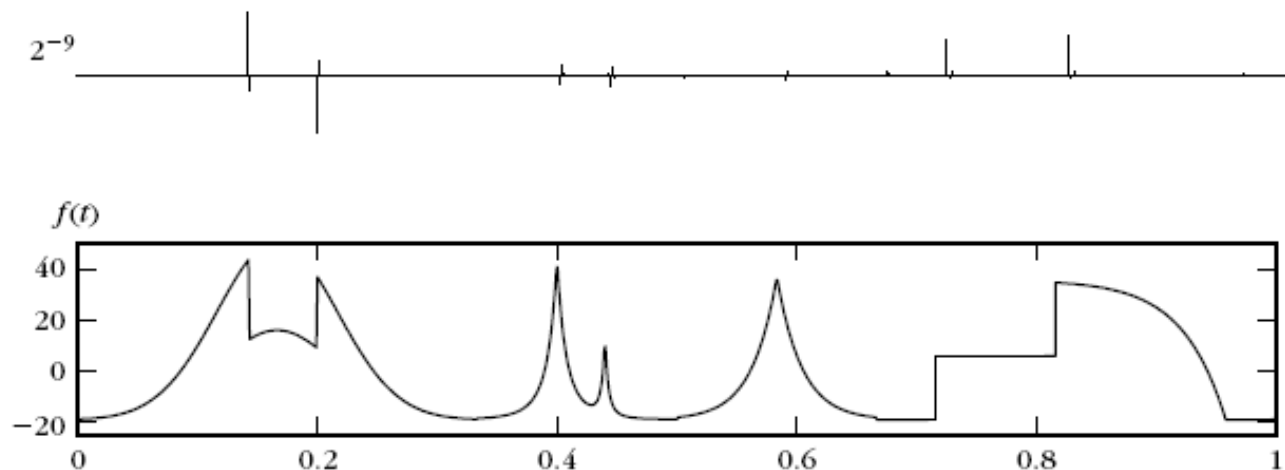
$$\varphi(t) : \exists p_n[k] : t^n = \sum_{k \in \mathbb{Z}} p_n[k] \varphi(t-k) \quad n = 0 \dots L-1$$

- Thus any polynomial of degree  $n$  up to  $L-1$  can be represented as a linear combination of scaling functions or, equivalently, the scaling function reproduces the polynomials up to degree  $L-1$ .
- There are other two S-F conditions but we focus on these two.

## Wavelets' *killing* property

Let us now prove that (4) implies (1). Since  $\psi$  is orthogonal to  $\{\phi(t - n)\}_{n \in \mathbb{Z}}$ , it is also orthogonal to the polynomials  $q_k$  for  $0 \leq k < p$ . This family of polynomials is a basis of the space of polynomials of degree at most  $p - 1$ . Thus,  $\psi$  is orthogonal to any polynomial of degree  $p - 1$  and in particular to  $t^k$  for  $0 \leq k < p$ . This means that  $\psi$  has  $p$  vanishing moments.

**A wavelet with  $p$  vanishing moments kills polynomials up to degree  $p$**



## Wavelet properties: support

- Support
  - The larger the support, the more the singularities will spread along scales: it should be **as short as possible**
  - BUT a wavelet with  $p$  vanishing moments will have a support at least  $2p-1$  -> trade-off
- **Proposition 7.2: Compact Support.** The scaling function has a compact support if and only if  $h$  has a compact support and their supports are equal. If the support of  $h$  and  $\varphi$  is  $[N_1, N_2]$ , then the support of  $\psi$  is  $[(N_1 - N_2 + 1)/2, (N_2 - N_1 + 1)/2]$ .

$$\hat{\Psi}(\omega) = \frac{1}{\sqrt{2}} \hat{g}\left(\frac{\omega}{2}\right) \hat{\Phi}\left(\frac{\omega}{2}\right)$$

$$\hat{g}(\omega) = e^{-j\omega} \hat{h}^*(\omega + \pi)$$



Product in Fourier  $\rightarrow$  convolution in time  
 $g[n]$  has the same support of  $h[n]$

The relation between the supports of the wavelet and the basis function comes from the properties of the convolution applied to the shrunk function (the support of  $\psi(t/2)$  is the same as that of  $\varphi(t)$  thus the support of  $\psi(t)$ , that is a shrunk version, is the half.

## Proof

*Proof*<sup>1</sup>. If  $\phi$  has a compact support, since

$$h[n] = \frac{1}{\sqrt{2}} \left\langle \phi\left(\frac{t}{2}\right), \phi(t-n) \right\rangle,$$

we derive that  $h$  also has a compact support. Conversely, the scaling function satisfies

$$\frac{1}{\sqrt{2}} \phi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} h[n] \phi(t-n). \quad (7.79)$$

If  $h$  has a compact support then one can prove [144] that  $\phi$  has a compact support. The proof is not reproduced here.

To relate the support of  $\phi$  and  $h$ , we suppose that  $h[n]$  is non-zero for  $N_1 \leq n \leq N_2$  and that  $\phi$  has a compact support  $[K_1, K_2]$ . The support of  $\phi(t/2)$  is  $[2K_1, 2K_2]$ . The sum at the right of (7.79) is a function whose support is  $[N_1 + K_1, N_2 + K_2]$ . The equality proves that the support of  $\phi$  is  $[K_1, K_2] = [N_1, N_2]$ .

## Support of the wavelet

Let us recall from (7.73) and (7.72) that

$$\frac{1}{\sqrt{2}} \psi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} g[n] \phi(t-n) = \sum_{n=-\infty}^{+\infty} (-1)^{1-n} h[1-n] \phi(t-n).$$

If the supports of  $\phi$  and  $h$  are equal to  $[N_1, N_2]$ , the sum in the right-hand side has a support equal to  $[N_1 - N_2 + 1, N_2 - N_1 + 1]$ . Hence  $\psi$  has a support equal to  $[(N_1 - N_2 + 1)/2, (N_2 - N_1 + 1)/2]$ . ■

If  $h$  has a finite impulse response in  $[N_1, N_2]$ , Proposition 7.2 proves that  $\psi$  has a support of size  $N_2 - N_1$  centered at  $1/2$ . To minimize the size of the support, we must synthesize conjugate mirror filters with as few non-zero coefficients as possible.



## Wavelets' properties: regularity

- Support
  - To minimize the size of the support of the wavelet, we must synthesize conjugate mirror filters with *as few nonzero coefficients as possible*
  - However, the constraints imposed on orthogonal wavelets imply that if *the wavelet* has  $p$  vanishing moments, then its support is at least of size  $2p-1$  → **trade off**
  - **Daubechies wavelets** are optimal in the sense that they have a **minimum size support for a given number of vanishing moments**
    - If  $f$  has **few isolated singularities** and is very regular between singularities, we must choose a wavelet with **many** vanishing moments to produce a large number of small wavelet coefficients  $\langle f, \psi_{j,n} \rangle$ . If the density of singularities increases, it might be better to decrease the size of its support at the cost of reducing the number of vanishing moments. Indeed, **wavelets that overlap the singularities create high-amplitude coefficients**.
- Regularity
  - The regularity or *smoothness* has mostly a cosmetic influence on the error introduced by *quantizing or thresholding* the coefficients. Such operation introduces a noise which is less visible if it is smooth. Better quality is reached with smoother wavelets
    - The Haar wavelet is not a good choice

# Wavelet families

## Orthogonal and compactly supported wavelets

- Wavelets: Daubechies (dbN), symlets (symN), coiflets (coifN).
- General properties:
  - -  $\phi$  exists and the analysis is orthogonal.
  - -  $\psi$  and  $\phi$  are compactly supported.
  - -  $\psi$  has a given number of vanishing moments.
- Possible analysis:
  - - continuous transform.
  - - discrete transform using FWT.
- Main nice properties: support, vanishing moments, FIR filters.
- Main difficulties: poor regularity.
- Specific properties:
  - For dbN : asymmetry
  - For symN : near symmetry
  - For coifN: near symmetry and  $\phi$  as  $\psi$ , has also vanishing moments.

# Daubechies compactly supported wavelets

## 7.2.3 Daubechies Compactly Supported Wavelets

Daubechies wavelets have a support of minimum size for any given number  $p$  of vanishing moments. Theorem 7.5 proves that wavelets of compact support are computed with finite impulse-response conjugate mirror filters  $h$ . We consider real causal filters  $h[n]$ , which implies that  $\hat{h}$  is a trigonometric polynomial:

$$\hat{h}(\omega) = \sum_{n=0}^{N-1} h[n] e^{-in\omega}.$$

To ensure that  $\psi$  has  $p$  vanishing moments, Theorem 7.4 shows that  $\hat{h}$  must have a zero of order  $p$  at  $\omega = \pi$ . To construct a trigonometric polynomial of minimal size, we factor  $(1 + e^{-i\omega})^p$ , which is a minimum-size polynomial having  $p$  zeros at  $\omega = \pi$ :

$$\hat{h}(\omega) = \sqrt{2} \left( \frac{1 + e^{-i\omega}}{2} \right)^p R(e^{-i\omega}). \quad (7.91)$$

The difficulty is to design a polynomial  $R(e^{-i\omega})$  of minimum degree  $m$  such that  $\hat{h}$  satisfies

$$|\hat{h}(\omega)|^2 + |\hat{h}(\omega + \pi)|^2 = 2. \quad (7.92)$$

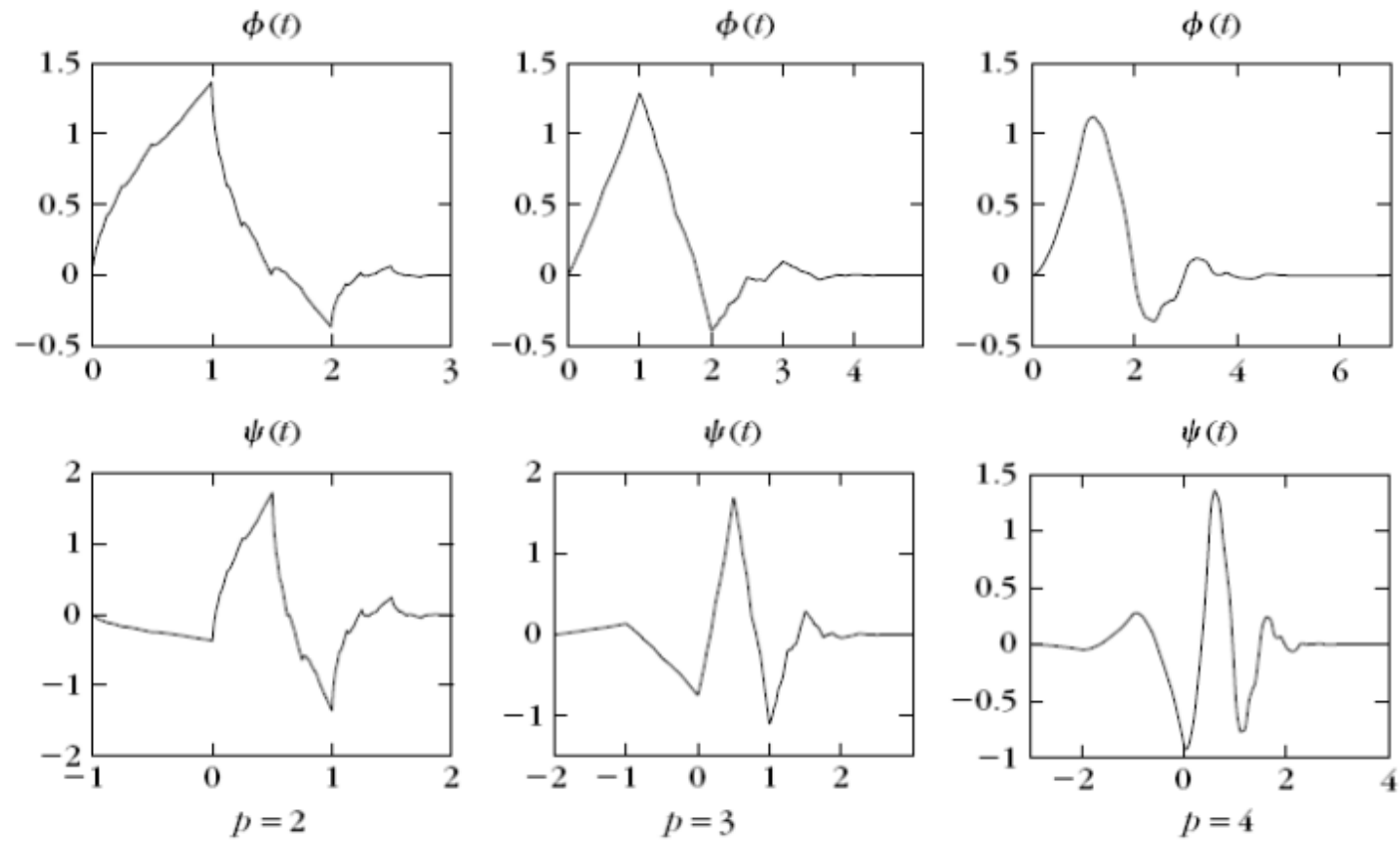
As a result,  $h$  has  $N = m + p + 1$  nonzero coefficients. Theorem 7.7 by Daubechies [194] proves that the minimum degree of  $R$  is  $m = p - 1$ .

## Daubechies compactly supported wavelets

- **Theorem 7.7: Daubechies.** A real conjugate mirror filter  $h$ , such that  $\hat{h}(\omega)$  has  $p$  zeroes at  $\pi$ , has at least  $2p$  nonzero coefficients. Daubechies filters have  $2p$  nonzero coefficients.
- **Theorem 7.9: Daubechies.** If  $\psi$  is a wavelet with  $p$  vanishing moments that generates an orthonormal basis of  $L^2(\mathbb{R})$ , then it has a support of size larger than or equal to  $2p+1$ .

A Daubechies wavelet has a *minimum-size support* equal to  $[-p+1, p]$ . The support of the corresponding scaling function is  $[0, 2p-1]$ .

## Daubechies wavelets: example



**FIGURE 7.10**

Daubechies scaling function  $\phi$  and wavelet  $\psi$  with  $p$  vanishing moments.

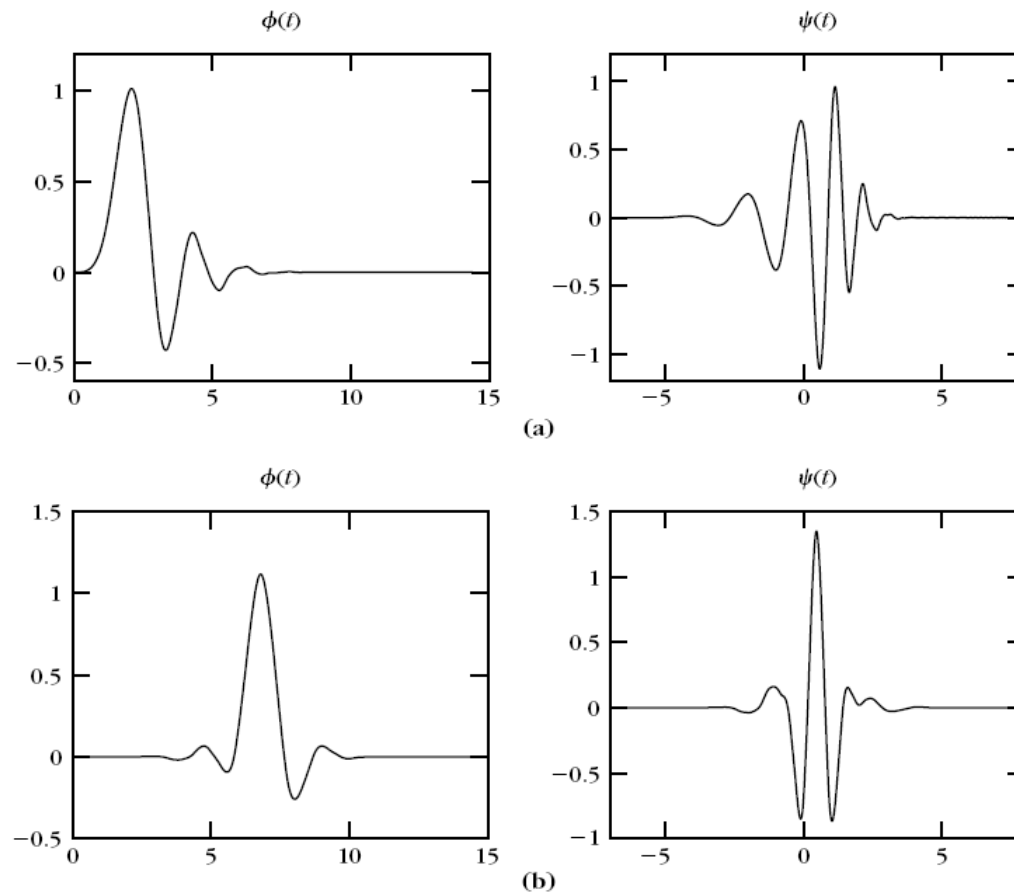
# Symlets

## *Symmlets*

Daubechies wavelets are very asymmetric because they are constructed by selecting the minimum-phase square root of  $Q(e^{-l\omega})$  in (7.97). One can show [51] that filters corresponding to a minimum-phase square root have their energy optimally concentrated near the starting point of their support. Thus, they are highly nonsymmetric, which yields very asymmetric wavelets.

To obtain a symmetric or antisymmetric wavelet, the filter  $h$  must be symmetric or antisymmetric with respect to the center of its support, which means that  $\hat{h}(\omega)$  has a linear complex phase. Daubechies proved [194] that the Haar filter is the only real compactly supported conjugate mirror filter that has a linear phase. The Daubechies *symmlet* filters are obtained by optimizing the choice of the square root  $R(e^{-l\omega})$  of  $Q(e^{-l\omega})$  to obtain an almost linear phase. The resulting wavelets still have a minimum support  $[-p + 1, p]$  with  $p$  vanishing moments, but they are more symmetric, as illustrated by Figure 7.11 for  $p = 8$ . The coefficients of the symmlet filters are in WAVELAB. Complex conjugate mirror filters with a compact support and a linear phase can be constructed [352], but they produce complex wavelet coefficients that have real and imaginary parts that are redundant when the signal is real.

# Dubechies versus Symlets



**FIGURE 7.11**

Daubechies (a) and symmet (b) scaling functions and wavelets with  $p = 8$  vanishing moments.



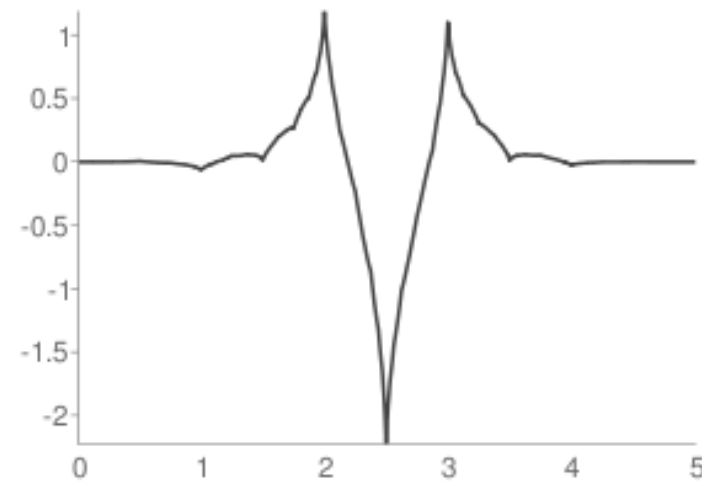
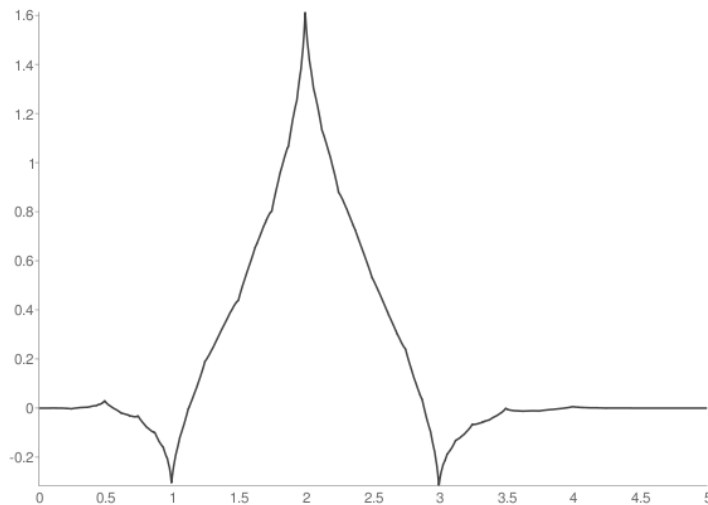
# Coiflets

## *Coiflets*

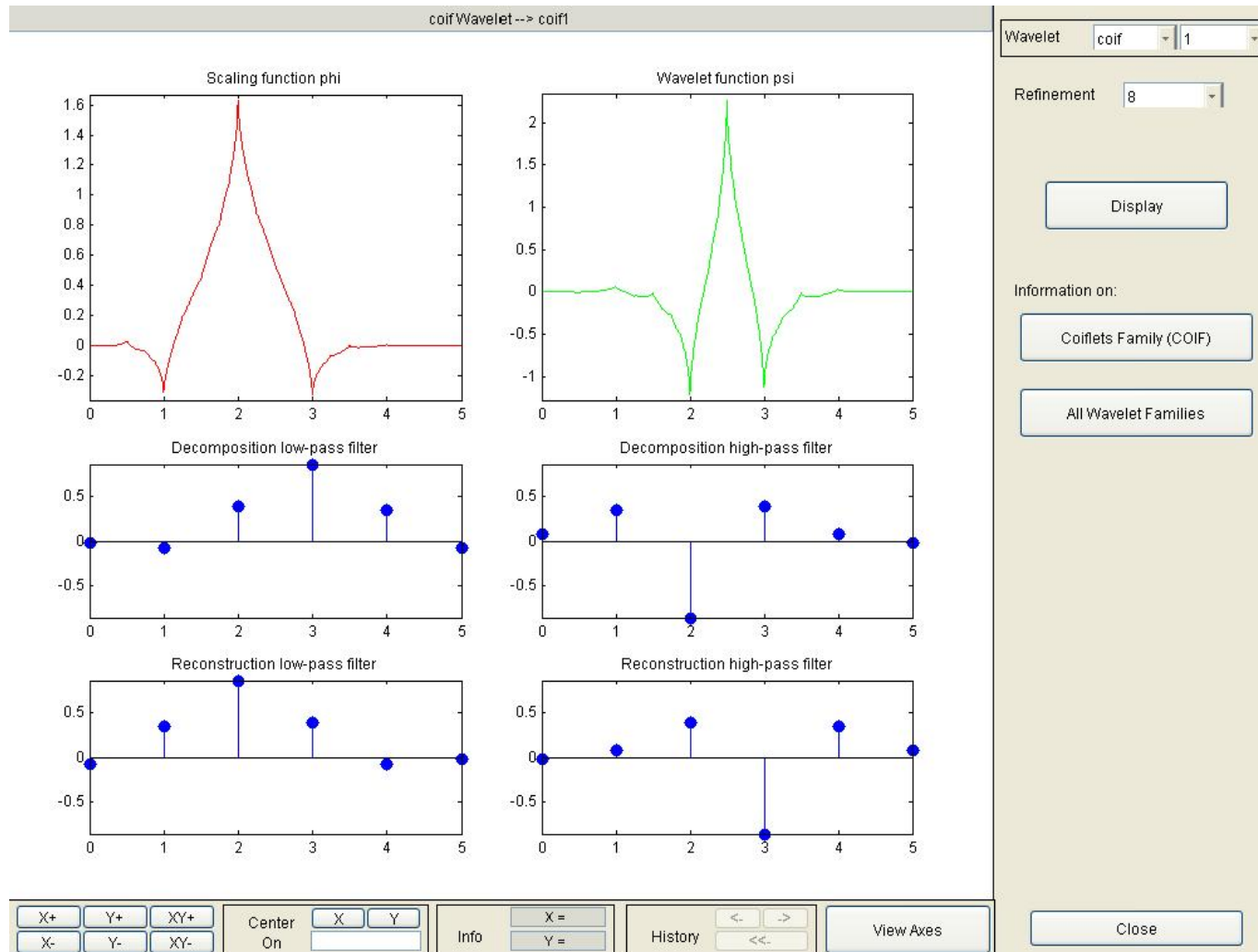
For an application in numerical analysis, Coifman asked Daubechies [194] to construct a family of wavelets  $\psi$  that have  $p$  vanishing moments and a minimum-size support, with scaling functions that also satisfy

$$\int_{-\infty}^{+\infty} \phi(t) dt = 1 \quad \text{and} \quad \int_{-\infty}^{+\infty} t^k \phi(t) dt = 0 \quad \text{for } 1 \leq k < p. \quad (7.99)$$

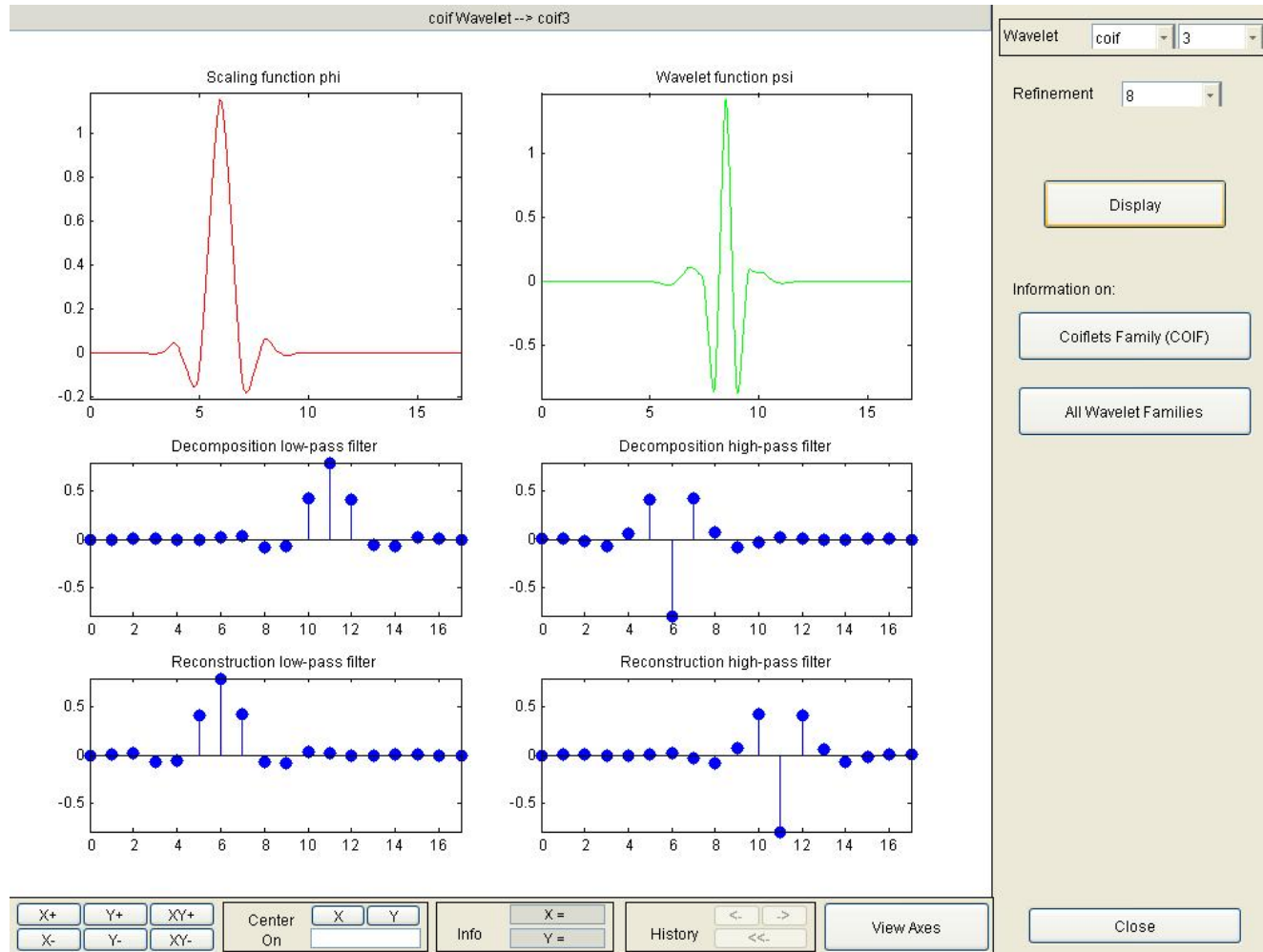
$p=1$



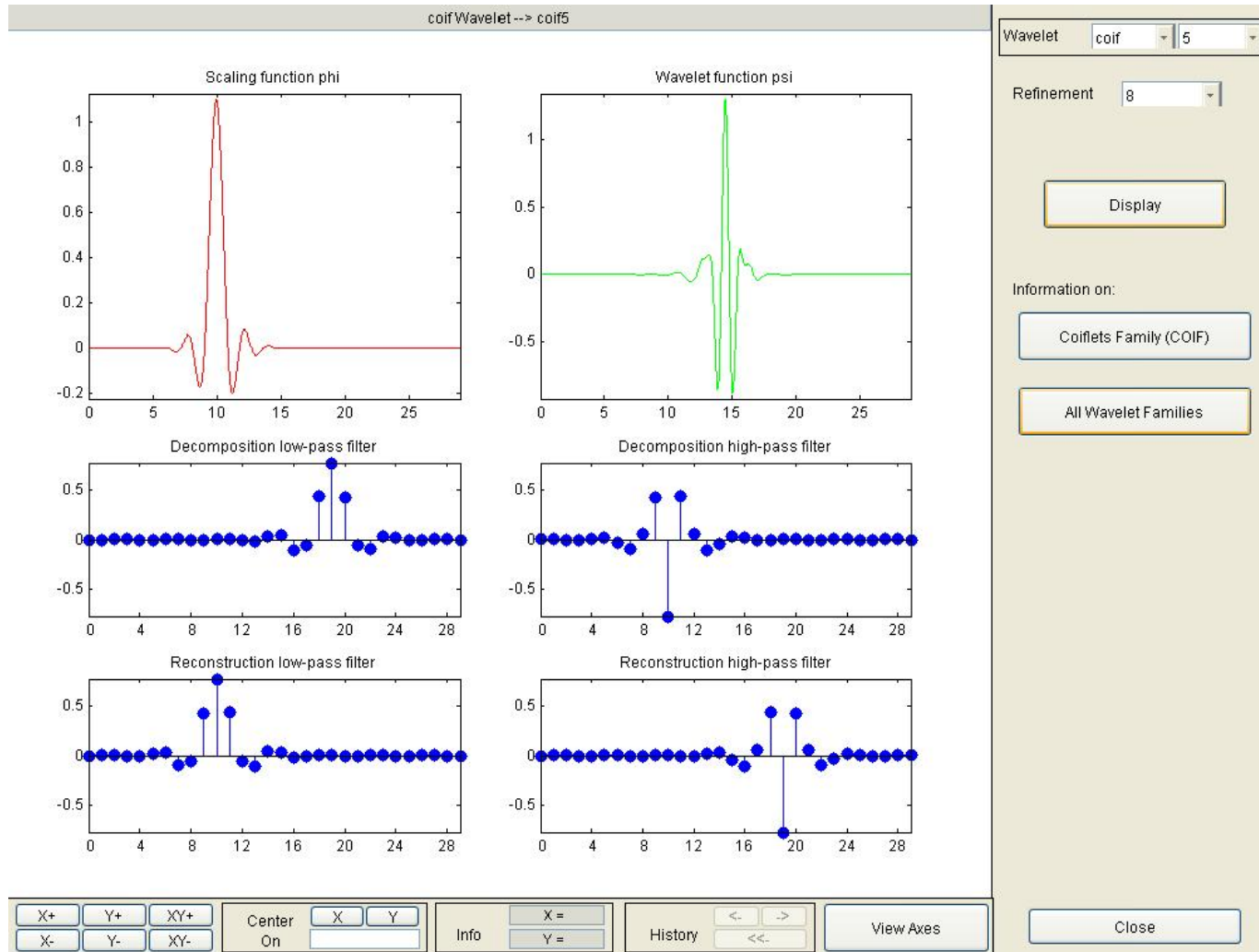
# Coiflets, order=1



# Coiflets, order=3



# Coiflets:order=5



# Biorthogonal wavelets

- Wavelets: B-splines biorthogonal wavelets (biorNr.Nd and rbioNr.Nd).
- Properties:
  - $\phi$  functions exist and the analysis is biorthogonal.
  - $\psi$  and  $\phi$  both for decomposition and reconstruction
  - are compactly supported.
  - $\phi$  and  $\psi$  for decomposition have vanishing moments.
  - $\psi$  and  $\phi$  for reconstruction have known regularity.
- Possible analysis:
  - continuous transform.
  - discrete transform using FWT.
- Main nice properties: symmetry with FIR filters, desirable properties for decomposition and reconstruction are split and nice allocation is possible.
- Main difficulties: orthogonality is lost.

# Infinitely regular wavelets

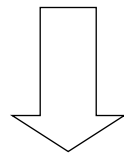
- Wavelets: meyer (meyr).
- Properties:
  - $\phi$  exists and the analysis is orthogonal
  - $\psi$  and  $\phi$  are indefinitely derivable.
  - $\psi$  and  $\phi$  are not compactly supported.
- Possible analysis:
  - continuous transform.
  - discrete transform but with non FIR filters.
- Main nice properties: symmetry, infinite regularity.
- Main difficulties: fast algorithm unavailable.
- Wavelets: discrete Meyer wavelet (dmey).
- Properties: FIR approximation of the Meyer wavelet
- Possible analysis:
  - continuous transform.
  - discrete transform.

## Shannon, Meyer Haar, Battle-Lemarié

- Shannon, Meyer, Haar, and Battle-Lemarié Wavelets
  - All derived from the general formula

$$\hat{\Psi}(\omega) = \frac{1}{\sqrt{2}} \hat{g}\left(\frac{\omega}{2}\right) \hat{\Phi}\left(\frac{\omega}{2}\right) \quad \hat{g}(\omega) = e^{-j\omega} \hat{h}^*(\omega + \pi)$$

$$\hat{\Psi}(2\omega) = \frac{1}{\sqrt{2}} \hat{g}(\omega) \hat{\Phi}(\omega) = e^{-i\omega} \hat{h}^*(\omega + \pi) \hat{\Phi}(\omega)$$



$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2}} \hat{g}\left(\frac{\omega}{2}\right) \hat{\phi}\left(\frac{\omega}{2}\right) = \frac{1}{\sqrt{2}} \exp\left(\frac{-i\omega}{2}\right) \hat{h}^*\left(\frac{\omega}{2} + \pi\right) \hat{\phi}\left(\frac{\omega}{2}\right). \quad (7.82)$$

## Shannon wavelets

### *Shannon Wavelet*

The Shannon wavelet is constructed from the Shannon multiresolution approximation, which approximates functions by their restriction to low-frequency intervals. It corresponds to  $\hat{\phi} = \mathbf{1}_{[-\pi, \pi]}$  and  $\hat{h}(\omega) = \sqrt{2} \mathbf{1}_{[-\pi/2, \pi/2]}(\omega)$  for  $\omega \in [-\pi, \pi]$ . We derive from (7.82) that

$$\hat{\psi}(\omega) = \begin{cases} \exp(-i\omega/2) & \text{if } \omega \in [-2\pi, -\pi] \cup [\pi, 2\pi] \\ 0 & \text{otherwise,} \end{cases} \quad (7.83)$$

and thus,

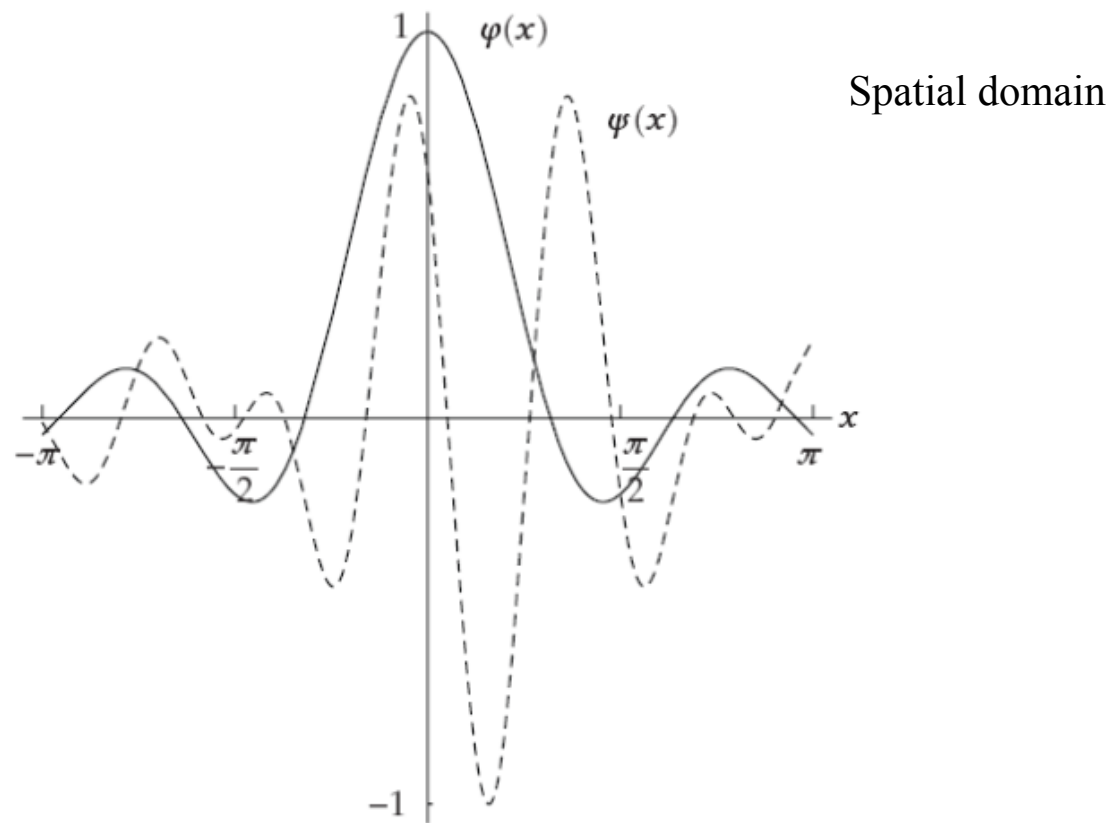
$$\text{Real SW} \quad \psi(t) = \frac{\sin 2\pi(t - 1/2)}{2\pi(t - 1/2)} - \frac{\sin \pi(t - 1/2)}{\pi(t - 1/2)}.$$

This wavelet is  $\mathbf{C}^\infty$  but has a slow asymptotic time decay. Since  $\hat{\psi}(\omega)$  is zero in the neighborhood of  $\omega = 0$ , all its derivatives are zero at  $\omega = 0$ . Thus, Theorem 7.4 implies that  $\psi$  has an infinite number of vanishing moments.

Since  $\hat{\psi}(\omega)$  has a compact support we know that  $\psi(t)$  is  $\mathbf{C}^\infty$ . However,  $|\psi(t)|$  decays only like  $|t|^{-1}$  at infinity because  $\hat{\psi}(\omega)$  is discontinuous at  $\pm\pi$  and  $\pm 2\pi$ .



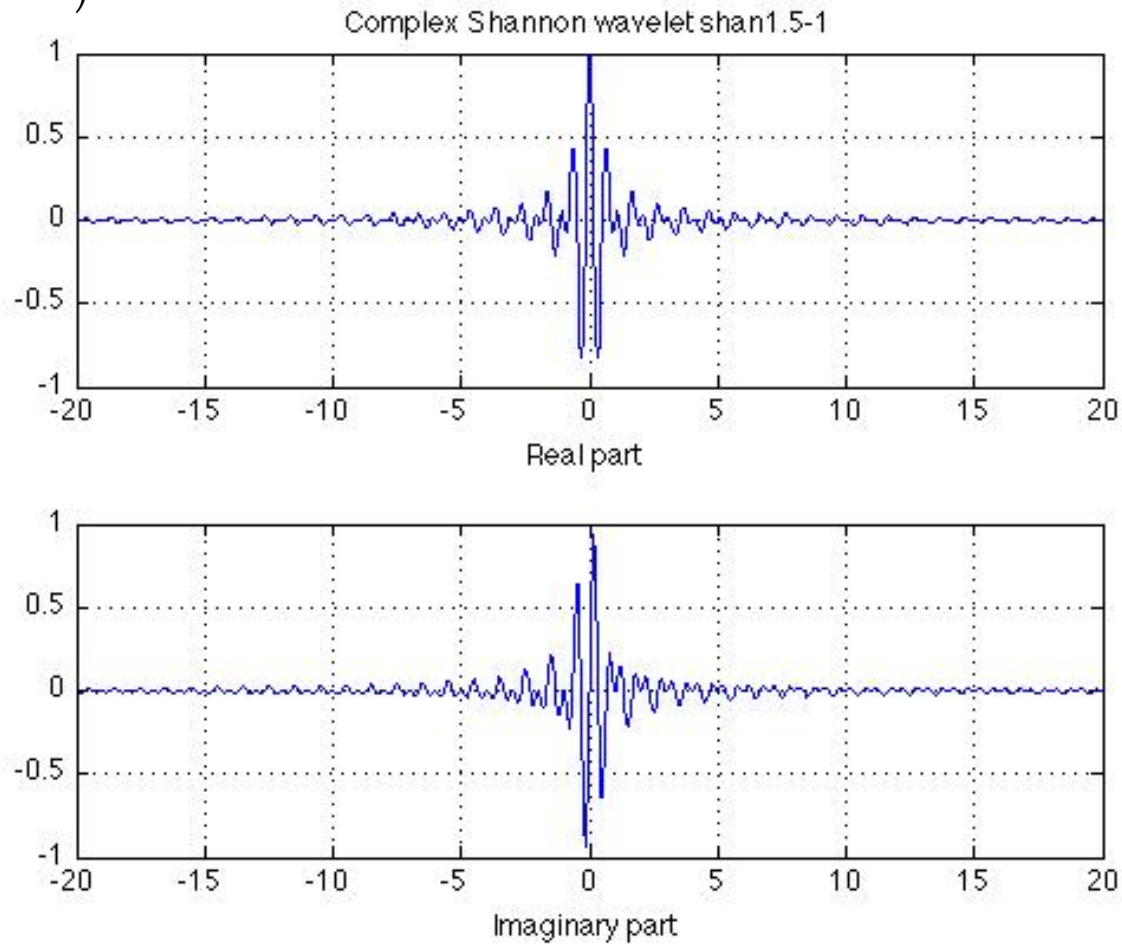
## Real Shannon wavelets



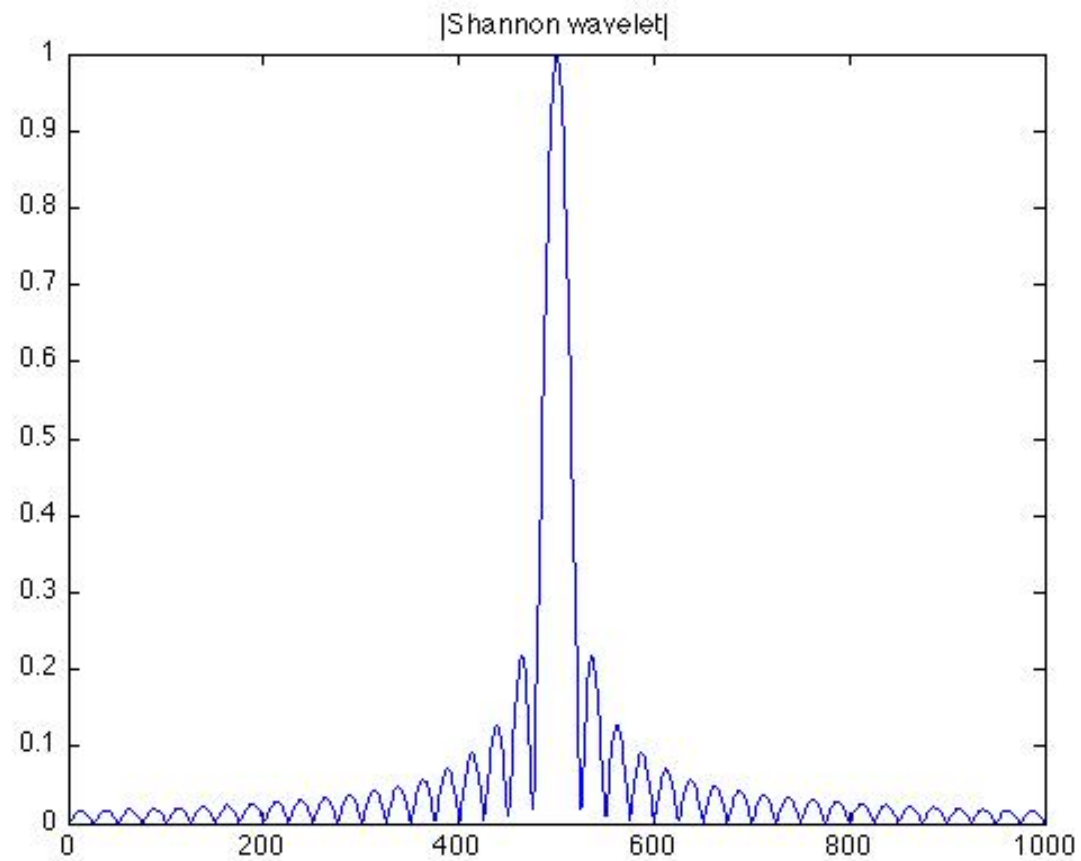
Shannon scaling function (continuous) and wavelet (dashed) lines.

# Complex Shannon wavelet

$$\psi(t) = \frac{\sin(t)}{t} \exp(-j2\pi t)$$



# Shannon wavelet



# Meyer wavelets

## *Meyer Wavelets*

A Meyer wavelet [375] is a frequency band-limited function that has a Fourier transform that is smooth, unlike the Fourier transform of the Shannon wavelet. This smoothness provides a much faster asymptotic decay in time. These wavelets are constructed with conjugate mirror filters  $\hat{h}(\omega)$  that are  $\mathbf{C}^n$  and satisfy

$$\hat{h}(\omega) = \begin{cases} \sqrt{2} & \text{if } \omega \in [-\pi/3, \pi/3] \\ 0 & \text{if } \omega \in [-\pi, -2\pi/3] \cup [2\pi/3, \pi]. \end{cases} \quad (7.84)$$

The only degree of freedom is the behavior of  $\hat{h}(\omega)$  in the transition bands  $[-2\pi/3, -\pi/3] \cup [\pi/3, 2\pi/3]$ . It must satisfy the quadrature condition

$$|\hat{h}(\omega)|^2 + |\hat{h}(\omega + \pi)|^2 = 2, \quad (7.85)$$

and to obtain  $\mathbf{C}^n$  junctions at  $|\omega| = \pi/3$  and  $|\omega| = 2\pi/3$ , the  $n$  first derivatives must vanish at these abscissa. One can construct such functions that are  $\mathbf{C}^\infty$ .

The scaling function  $\hat{\phi}(\omega) = \prod_{p=1}^{+\infty} 2^{-1/2} \hat{h}(2^{-p}\omega)$  has a compact support and one can verify that

$$\hat{\phi}(\omega) = \begin{cases} 2^{-1/2} \hat{h}(\omega/2) & \text{if } |\omega| \leq 4\pi/3 \\ 0 & \text{if } |\omega| > 4\pi/3. \end{cases} \quad (7.86)$$

## Meyer wavelets

The resulting wavelet (7.82) is

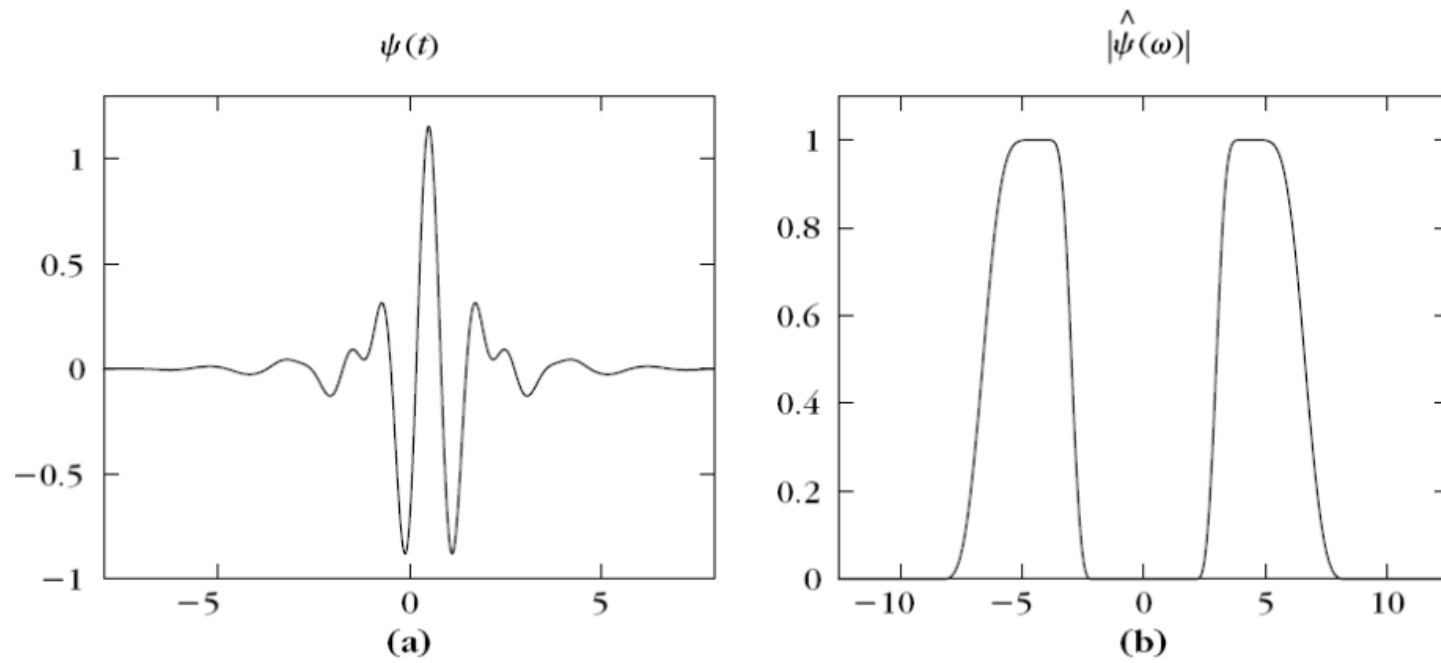
$$\hat{\psi}(\omega) = \begin{cases} 0 & \text{if } |\omega| \leq 2\pi/3 \\ 2^{-1/2} \hat{g}(\omega/2) & \text{if } 2\pi/3 \leq |\omega| \leq 4\pi/3 \\ 2^{-1/2} \exp(-i\omega/2) \hat{h}(\omega/4) & \text{if } 4\pi/3 \leq |\omega| \leq 8\pi/3 \\ 0 & \text{if } |\omega| > 8\pi/3. \end{cases} \quad (7.87)$$

The functions  $\phi$  and  $\psi$  are  $C^\infty$  because their Fourier transforms have a compact support. Since  $\hat{\psi}(\omega) = 0$  in the neighborhood of  $\omega = 0$ , all its derivatives are zero at  $\omega = 0$ , which proves that  $\psi$  has an infinite number of vanishing moments.

If  $\hat{h}$  is  $C^n$ , then  $\hat{\psi}$  and  $\hat{\phi}$  are also  $C^n$ . The discontinuities of the  $(n+1)^{\text{th}}$  derivative of  $\hat{h}$  are generally at the junction of the transition band  $|\omega| = \pi/3, 2\pi/3$ , in which case one can show that there exists  $A$  such that

$$|\phi(t)| \leq A (1 + |t|)^{-n-1} \quad \text{and} \quad |\psi(t)| \leq A (1 + |t|)^{-n-1}.$$

## Meyer wavelet: example



# Haar wavelets

## *Haar Wavelets*

The Haar basis is obtained with a multiresolution of piecewise constant functions. The scaling function is  $\phi = \mathbf{1}_{[0,1]}$ . The filter  $h[n]$  given in (7.46) has two nonzero coefficients equal to  $2^{-1/2}$  at  $n = 0$  and  $n = 1$ . Thus,

$$\frac{1}{\sqrt{2}} \psi \left( \frac{t}{2} \right) = \sum_{n=-\infty}^{+\infty} (-1)^{1-n} h[1-n] \phi(t-n) = \frac{1}{\sqrt{2}} \left( \phi(t-1) - \phi(t) \right),$$

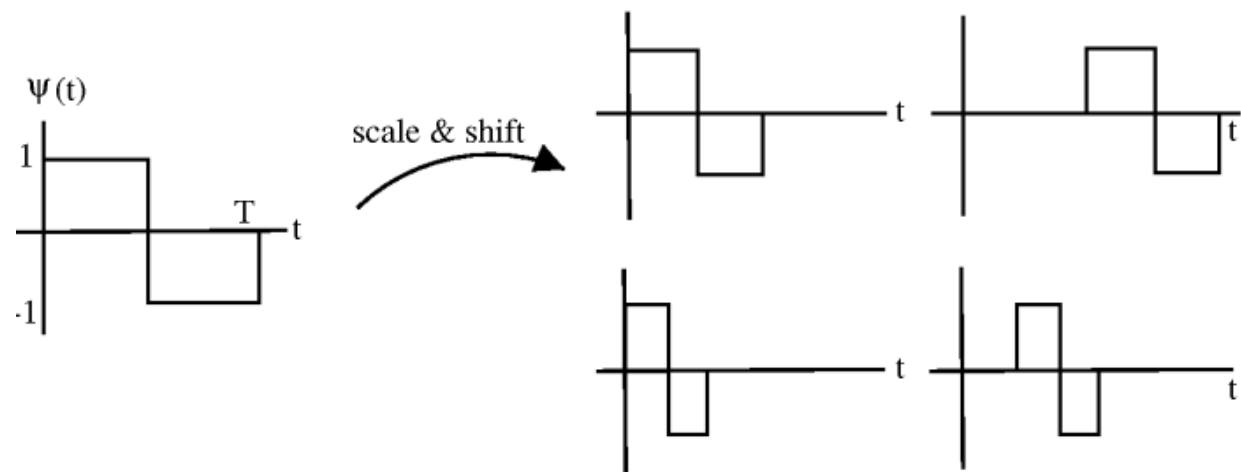
so

$$\psi(t) = \begin{cases} -1 & \text{if } 0 \leq t < 1/2 \\ 1 & \text{if } 1/2 \leq t < 1 \\ 0 & \text{otherwise.} \end{cases} \quad (7.90)$$

The Haar wavelet has the shortest support among all orthogonal wavelets. It is not well adapted to approximating smooth functions because it has only one vanishing moment.

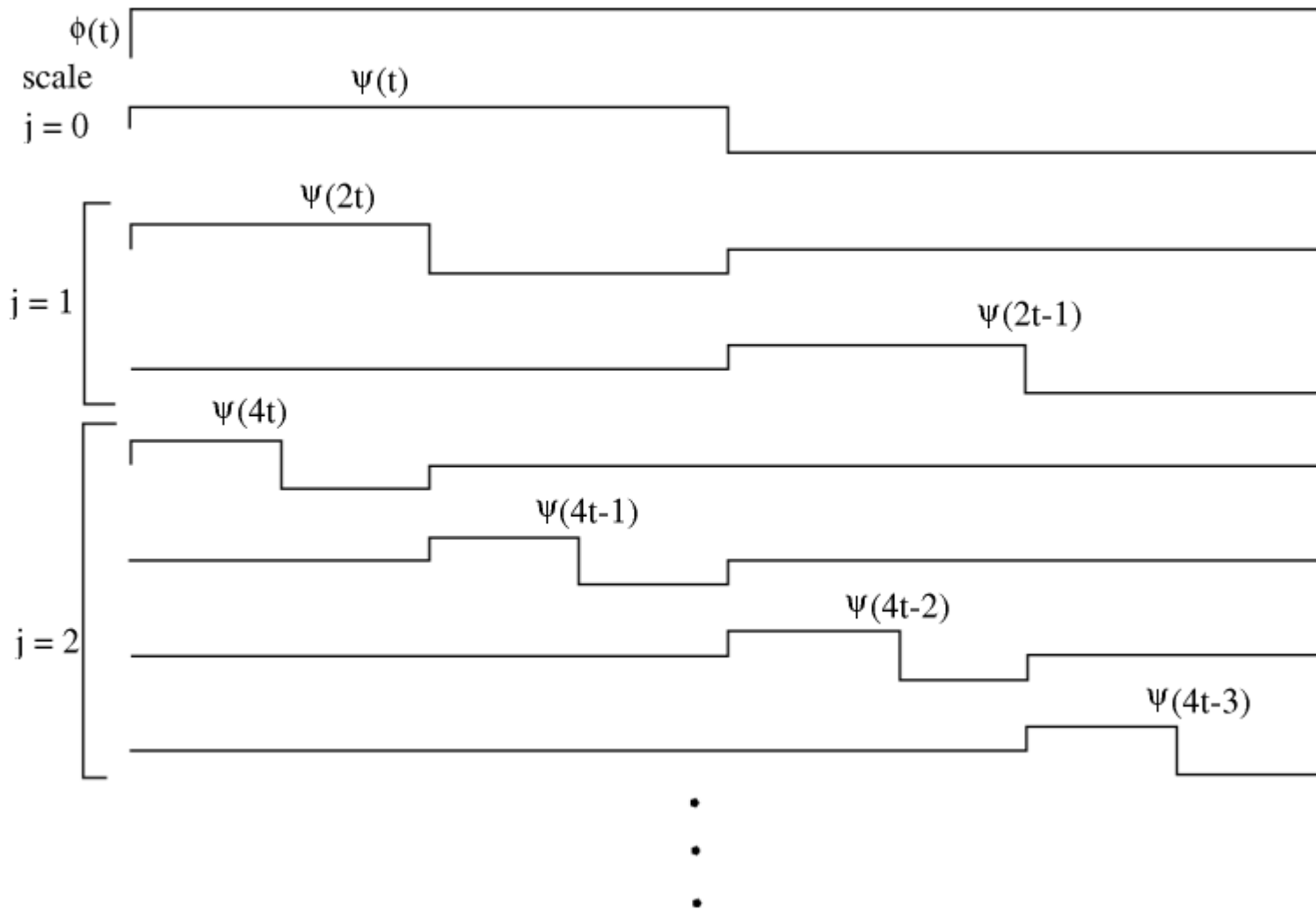
reminder: 
$$\frac{1}{\sqrt{2}} \psi \left( \frac{t}{2} \right) = \sum_{n=-\infty}^{+\infty} g[n] \phi(t-n) = \sum_{n=-\infty}^{+\infty} (-1)^{1-n} h[1-n] \phi(t-n).$$

# Haar wavelets





# Haar wavelets



## Battle-Lemarié wavelets

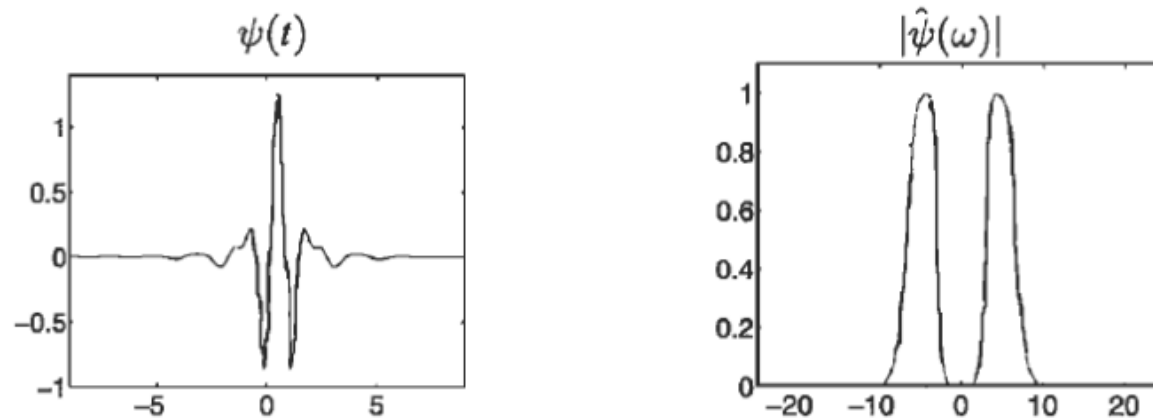
### ***Battle-Lemarié Wavelets***

Polynomial spline wavelets introduced by Battle [99] and Lemarié [345] are computed from spline multiresolution approximations. The expressions of  $\hat{\phi}(\omega)$  and  $\hat{h}(\omega)$  are given, respectively, by (7.18) and (7.48). For splines of degree  $m$ ,  $\hat{h}(\omega)$  and its first  $m$  derivatives are zero at  $\omega = \pi$ . Theorem 7.4 derives that  $\psi$  has  $m + 1$  vanishing moments. It follows from (7.82) that

$$\hat{\psi}(\omega) = \frac{\exp(-i\omega/2)}{\omega^{m+1}} \sqrt{\frac{S_{2m+2}(\omega/2 + \pi)}{S_{2m+2}(\omega) S_{2m+2}(\omega/2)}}.$$

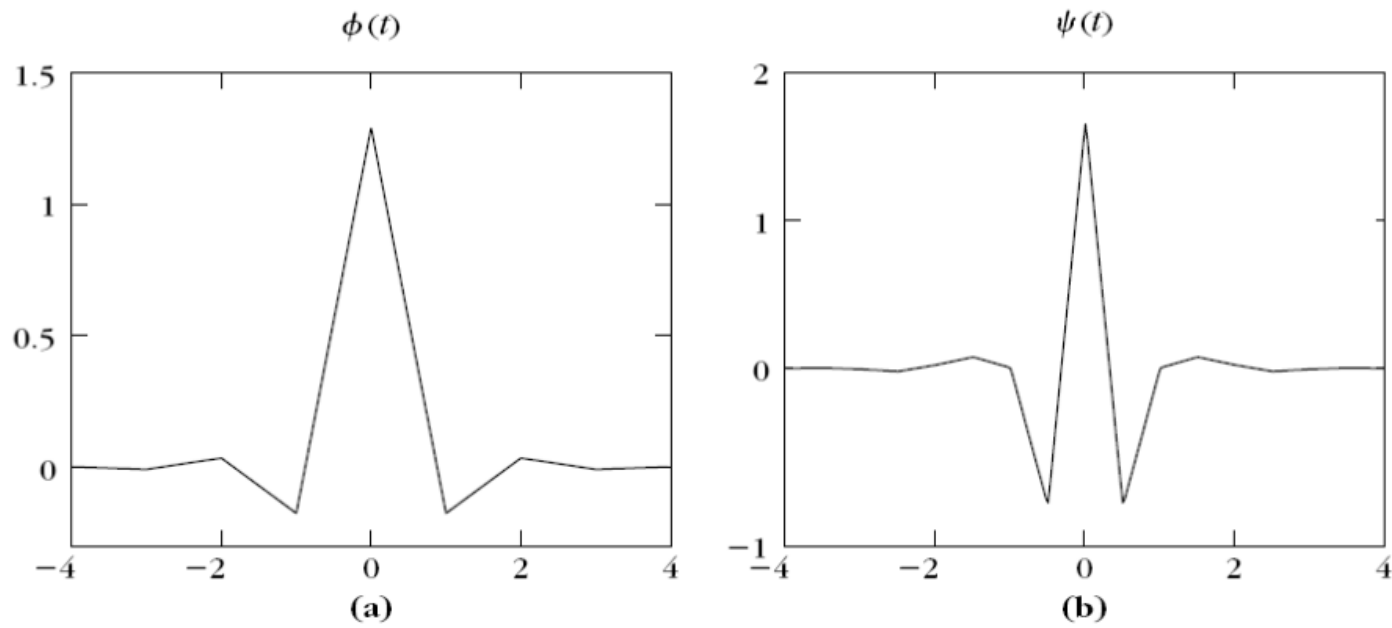
This wavelet  $\psi$  has an exponential decay. Since it is a polynomial spline of degree  $m$ , it is  $m - 1$  times continuously differentiable. Polynomial spline wavelets are less regular than Meyer wavelets but have faster time asymptotic decay. For  $m$  odd,  $\psi$  is symmetric about  $1/2$ . For  $m$  even, it is antisymmetric about  $1/2$ . Figure 7.5 gives the graph of the cubic spline wavelet  $\psi$  corresponding to  $m = 3$ . For  $m = 1$ , Figure 7.9 displays linear splines  $\phi$  and  $\psi$ . The properties of these wavelets are further studied in [15, 106, 164].

## Battle-Lemarié cubic bispline



**FIGURE 7.5** Battle-Lemarié cubic spline wavelet  $\psi$  and its Fourier transform modulus.

# Linear Battle-Lemarié



**FIGURE 7.9**

Linear spline Battle-Lemarié scaling function  $\phi$  (a) and wavelet  $\psi$  (b).

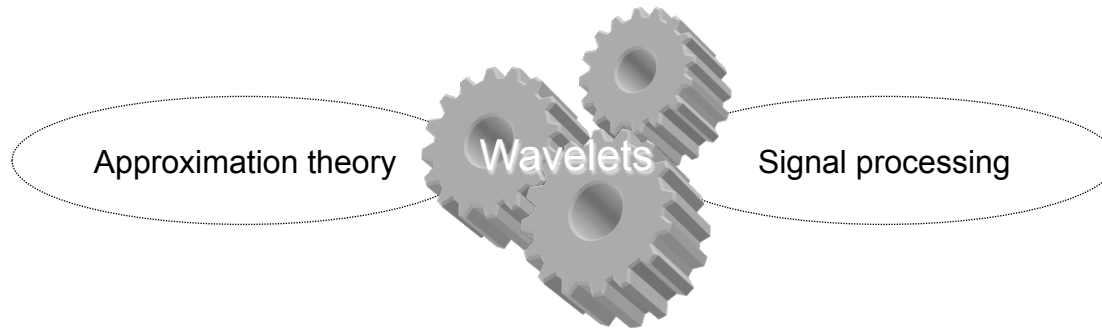
## Crude wavelets

- Wavelets: gaussian wavelets (gaus), morlet, mexican hat (mexihat).
- Properties: only minimal properties
  - **phi does not exist.**
  - the analysis is not orthogonal.
  - psi is not compactly supported.
  - the reconstruction property is not insured.
- Possible analysis:
  - continuous decomposition.
- Main nice properties: symmetry, psi has explicit expression.
- Main difficulties: fast algorithm and reconstruction unavailable.

## Complex wavelets

- Wavelets: Complex Gaussian wavelets (cgauN), complex Morlet wavelets (cmorFb-Fc), complex Shannon wavelets (shanFb-Fc), complex frequency B-spline wavelets (fbspM-Fb-Fc).
- Properties: only minimal properties
  - $\phi$  does not exist.
  - the analysis is not orthogonal
  - $\psi$  is not compactly supported
  - the reconstruction property is not insured.
- Possible analysis:
  - complex continuous decomposition.
- Main nice properties: symmetry,  $\psi$  has explicit expression.
- Main difficulties: fast algorithm and reconstruction unavailable.

# An approximation tour



- Linear approximation

- Projects the signal  $f$  over  $M$  vectors of the ortho-normal basis  $B$  which are chosen *a-priori* among the basis  $B$ , say the first  $M$

$$f_M = \sum_{n=0}^{M-1} \langle f, \phi_n \rangle \phi_n$$

- Approximation error  
$$\varepsilon[M] = \|f - f_M\|^2 = \sum_{n=M}^{+\infty} |\langle f, \phi_n \rangle|^2$$

choosing the first  $M$  vectors amounts to reconstruct  $f$  at a given resolution. The convergence properties similar as in the Fourier domain

- Non-linear approximations

- The  $M$  vectors are chosen *a posteriori*

$$f_M = \sum_{n \in I_M} \langle f, \phi_n \rangle \phi_n$$

Approximation error

$$\varepsilon[M] = \|f - f_M\|^2 = \sum_{n \in I_M} |\langle f, \phi_n \rangle|^2$$

The error can be minimized by choosing the vectors corresponding to the highest  $|\langle f, \phi_n \rangle|$

In wavelet basis this amounts to an *adaptive approximation grid* whose *resolution is locally increased where the signal is irregular!*

## Adaptive basis choice

- Instead of choosing the basis a-priori, one could choose the *best* basis, depending on the signal
- The basis is chosen to minimize the non linear approximation error of  $f$
- Same problem as the choice of the *optimal basis* for stimulus representation in visual perception
- The optimal basis could be chosen for *classes of signals*, considered as random processes
  - Gaussian processes → Karunen Loeve transform (KLT)
    - Diagonalization of the covariance matrix which removes the inter-dependencies among the samples and results in a set of independent coefficients (i.e. redundancy has been removed)
  - Other kind of processes → no golden rule
    - Images are not Gaussian and not stationary
    - In some cases wavelets do better



# Adaptive basis

- Wavelet packets
  - The subband tree is progressively split according to the optimization of a cost function (i.e. rate/distortion)
- Matching pursuit
  - Vectors are progressively selected from a dictionary, while optimizing the signal approximation at each step
- Key issue: a good basis should be able to provide a good description (approximation properties) of the signal while being concise (sparseness properties)
  - Classical approaches: approximation theory, information theory, estimation in noise...
  - Perception based approaches: bring humans into the loop

# Wavelet Packets

