

Multiresolution analysis

Analisi multirisoluzione

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The Fourier kingdom

- CTFT

- Continuous time signals

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t} dt$$

$$f(t) = \int_{-\infty}^{+\infty} F(\omega)e^{j\omega t} dt$$

- The amplitude $F(\omega)$, also called Fourier transform, of each sinusoidal wave $e^{-j\omega t}$ is equal to its correlation with f
- If $f(t)$ is uniformly regular, then its Fourier transform coefficients also have a fast decay when the frequency increases, so it can be easily approximated with few low-frequency Fourier coefficients.

The Fourier kingdom

- DTFT
 - Over discrete signals, the Fourier transform is a decomposition in a discrete orthogonal Fourier basis $\{e^{i2kn/N}\}_{0 \leq k < N}$ of C^N , which has properties similar to a Fourier transform on functions.
 - Its embedded structure leads to fast Fourier transform(FFT) algorithms, which compute discrete Fourier coefficients with $O(N \log N)$ instead of N^2 . This FFT algorithm is a cornerstone of discrete signal processing.
- The Fourier transform is unsuitable for representing transient phenomena
 - the support of $e^{-\omega it}$ covers the whole real line, so $\hat{f}(\omega)$ depends on the values $f(t)$ for all times $t \in \mathbb{R}$. This -global “mix” of information makes it difficult to analyze or represent any local property of $f(t)$ from $\hat{f}(\omega)$.
 - As long as we are satisfied with *linear time-invariant* operators or *uniformly regular signals*, the Fourier transform provides simple answers to most questions. Its richness makes it suitable for a wide range of applications such as signal transmissions or stationary signal processing. However, to represent a *transient* phenomenon—a word pronounced at a particular time, an apple located in the left corner of an image—the Fourier transform becomes a cumbersome tool that requires *many coefficients* to represent a *localized* event.

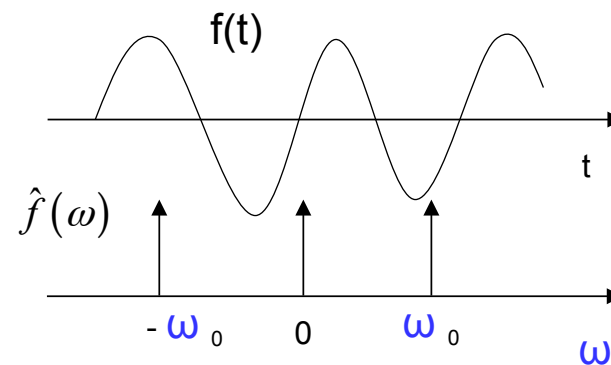
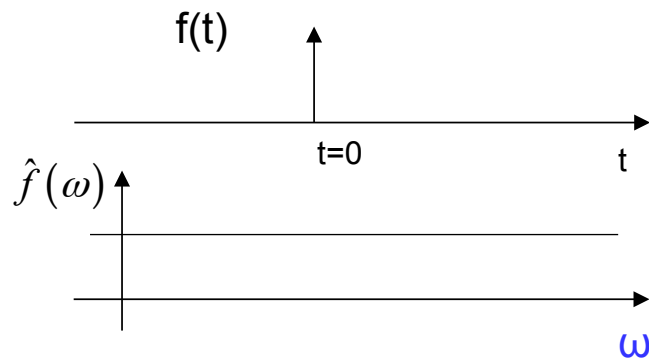
The Fourier kingdom

- The F-transform is not suitable for representing transient phenomena

- Intuition

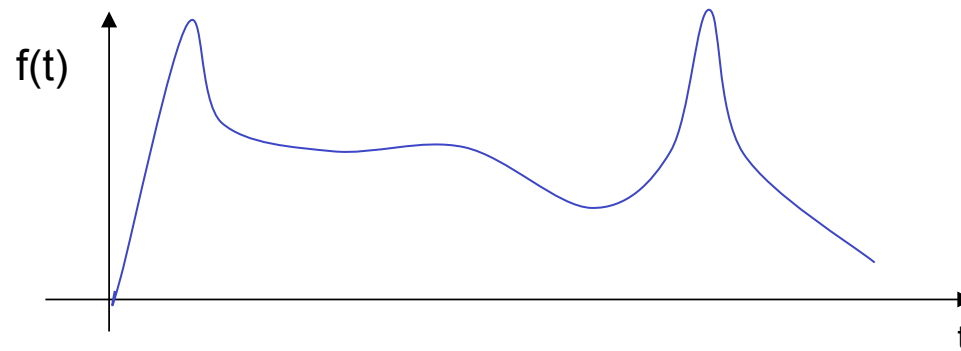
$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t} dt$$

- $F(\omega)$ depends on the values taken by $f(t)$ on the entire temporal axis, which is not suitable for analyzing local properties
- Need of a transformation which is well localized in *time and frequency*



The Fourier kingdom

- Transient phenomena

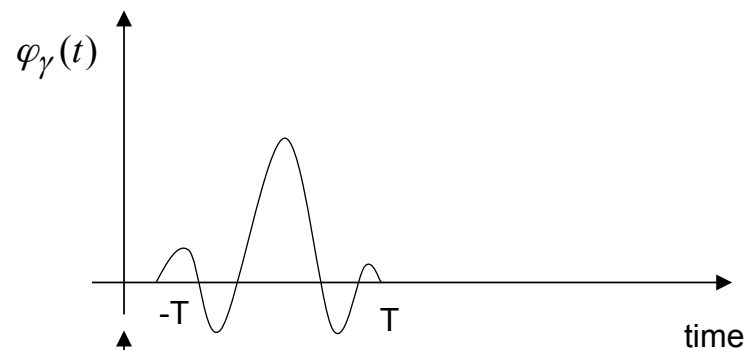


The two transients present in the signal contribute **differently** to the spectrum. The F-transform does not allow to characterize them **separately** to get a local description of the frequency content of the signal.

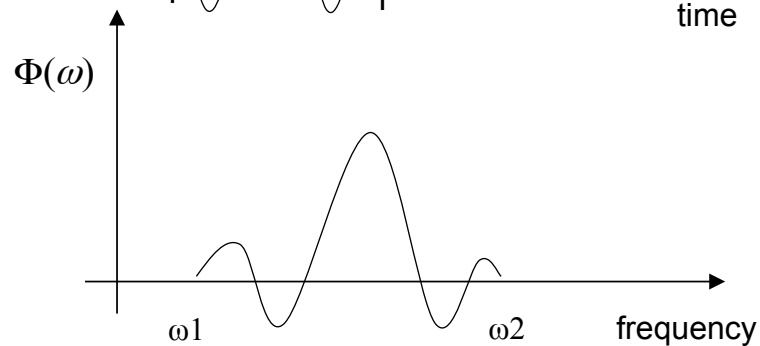
The basis functions of the FT are complex sinusoids, thus **$F(\omega)$ is a measure of the correlation of the signal $f(t)$ with the complex exponential at frequency ω** , which spreads over the whole frequency axis.

Time-frequency localization

- Time-frequency atoms: basis functions that are well localized in *both* time and frequency



$\langle f(t), \varphi_\gamma(t) \rangle$ only depends on the values of f in the neighborhood of T



$F(\omega)\Phi(\omega)$ only depends on the values of F in the neighborhood of ω

Windowed Fourier Transform

Windowed Fourier atoms were introduced in 1946 by Gabor to measure localized frequency components of sounds

Also *short time Fourier transform*

$$g_{u,\xi}(t) = e^{j\xi t} g(t-u)$$

$$\|g\| = 1 \rightarrow \|g_{u,\xi}\| = 1 \quad \forall(u, \xi)$$

$$Sf(u, \xi) = \langle f, g_{u,\xi} \rangle = \int_{-\infty}^{+\infty} f(t) g_{u,\xi}^*(t) dt = \int_{-\infty}^{+\infty} f(t) g(t-u) e^{-j\xi t} dt$$

The Fourier integral is *localized in the neighborhood of u by the window $g(t-u)$* .

The transform $Sf(u, \xi)$ depends only on the values of $f(t)$ and $f(\omega)$ in the **time and frequency neighborhoods** where the energy is concentrated, providing information on the behavior of the function **within a bounded time-frequency interval**.

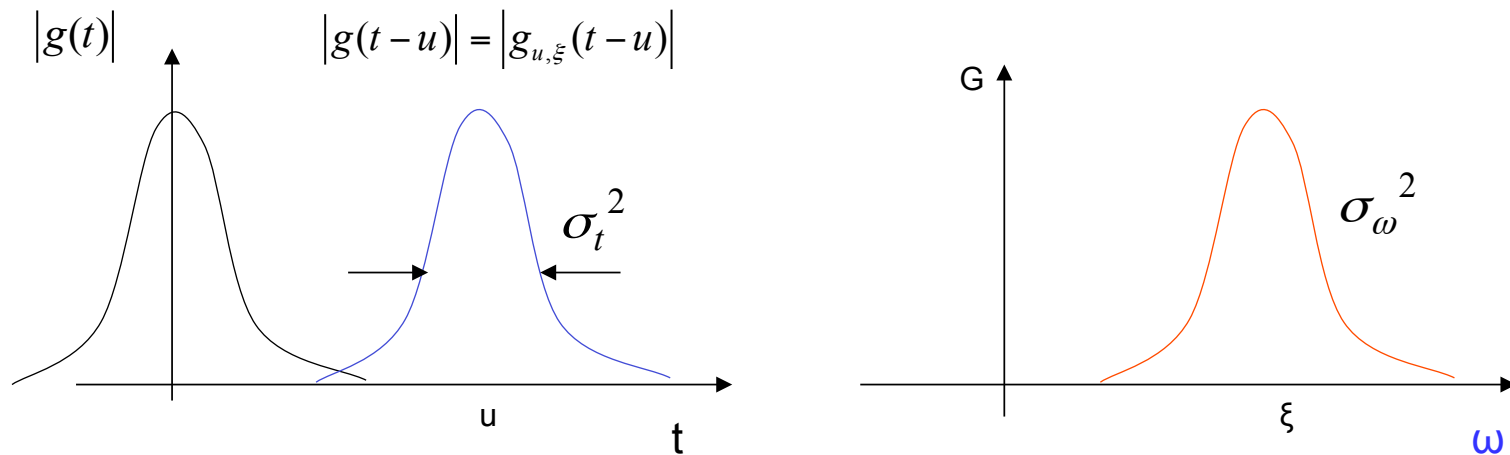
Wavelet genesis : the WT was designed following the same approach, with the goal of characterizing transient phenomena in signals by a mapping into the time/frequency domain.

Windowed Fourier Transform

$$g_{u,\xi}(t) = g(t-u)e^{j\xi t}$$

$$G_{u,\xi}(\omega) = G(\omega - \xi)e^{-ju(\omega - \xi)}$$

$$E = \int_{-\infty}^{+\infty} |g(t)|^2 dt = (\text{Plancherel}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |G(\omega)|^2 d\omega$$



Time-frequency atoms

- The time localization u and spread around u are defined as

$g_{u,\xi}(t)$ is centered in u

$$\|\sigma_t^2\| = \int_{-\infty}^{+\infty} |t-u|^2 |g_{u,\xi}(t)|^2 dt = \int_{-\infty}^{+\infty} t^2 |g(t)|^2 dt$$

interpreted as a probability distribution

- Similarly, the frequency localization ξ and the spread around ξ are defined as

$$\|\sigma_\omega^2\| = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\omega-\xi|^2 |G_{u,\xi}(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \omega^2 |G(\omega)|^2 d\omega$$

time spread around u

independent of u and ξ

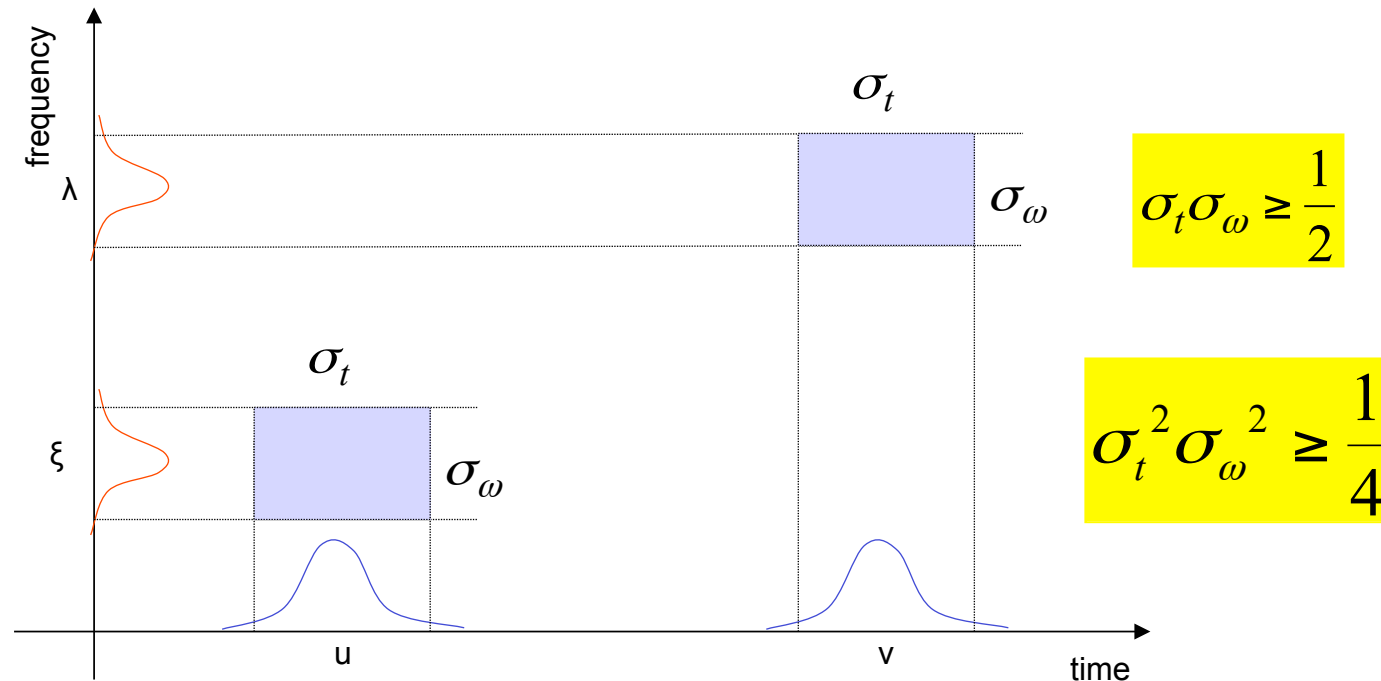
frequency spread around ξ

$g_{u,\xi}(t)$ corresponds to a Heisenberg box of surface $\sigma_t \sigma_\omega$

that is independent of u and ξ , hence the windowed Fourier transform has the same resolution across the time-frequency plan

Heisemberg boxes

- Uncertainty principle: the rectangle has a minimum surface that limits the joint time-frequency resolution



When g is a Gaussian the atoms are called Gabor functions. Since in this case the equality holds, these [minimize the area of the Heisenberg box](#), Gabor atoms are considered as [optimal for the time-frequency characterization of signals](#).

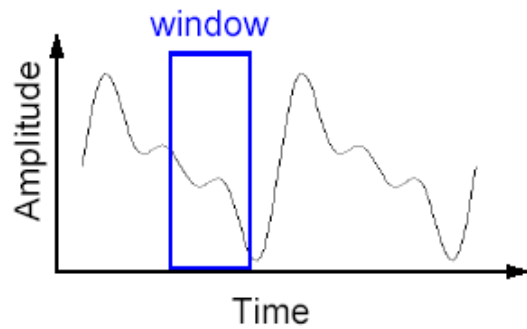
Windowed Fourier Transform

- It can be interpreted as a Fourier transform of f at the frequency ξ , localized by the window $g(t-u)$ in the neighborhood of u . This windowed Fourier transform is **highly redundant** and represents one-dimensional signals by a time-frequency image in (u, ξ) . It is thus necessary to understand how to select many fewer time frequency coefficients that represent the signal efficiently.
- A windowed Fourier transform decomposes signals over **waveforms that have the same time and frequency resolution**. *It is thus effective as long as the signal does not include structures having different time-frequency resolutions, some being very localized in time and others very localized in frequency.*
- Wavelets address this issue by **changing the time and frequency resolution**.

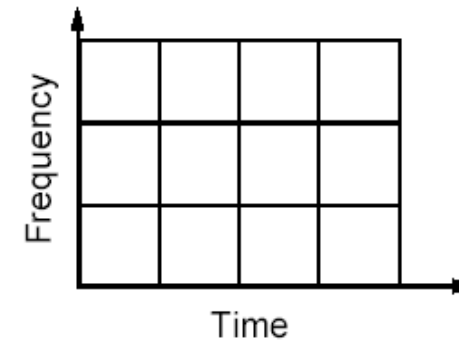
Short Time Fourier Transform

- The STFT (windowed FT) represents a sort of compromise between the time- and frequency-based views of a signal. It provides some information about both *when* and at *what frequencies* a signal event occurs.
- However, you can only obtain this information with limited *precision*, and that *precision is determined by the size of the window*
- While the STFT compromise between time and frequency information can be useful, the drawback is that once you choose a particular **size** for the time window, that window is the **same for all frequencies**
- Many signals require a more flexible approach, one where we can vary the window size to determine more accurately either time or frequency.

Windowed Fourier transform



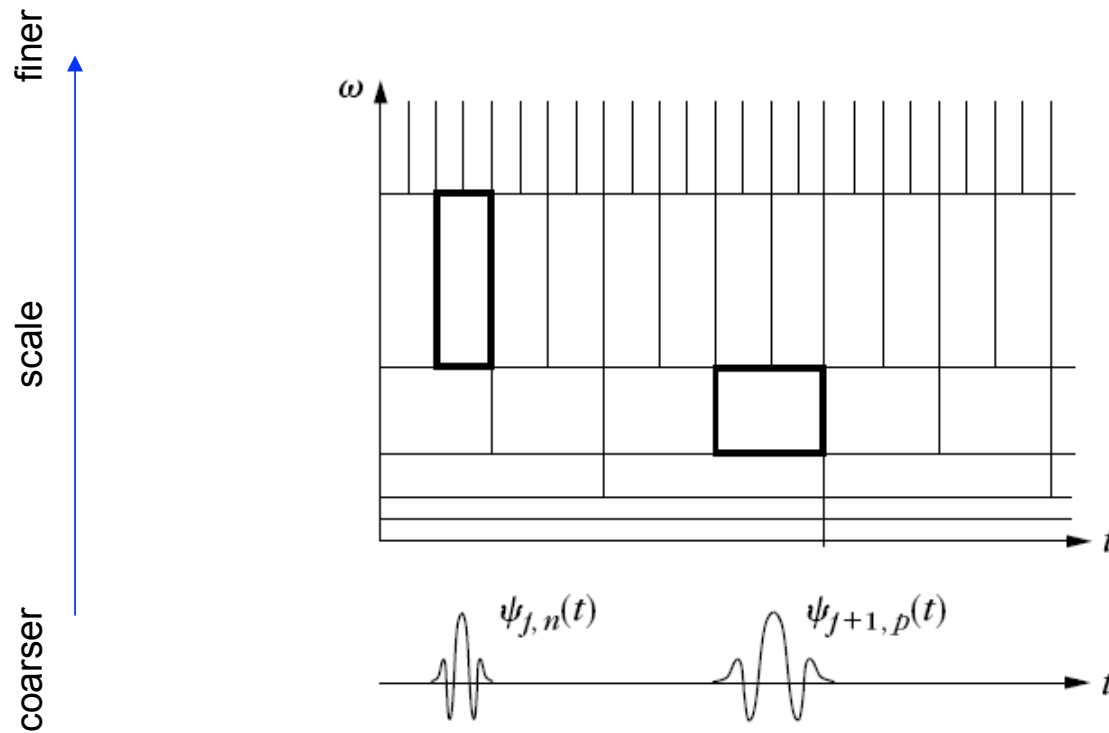
Short
Time
Fourier
Transform



Uniform tiling of the time-frequency plan

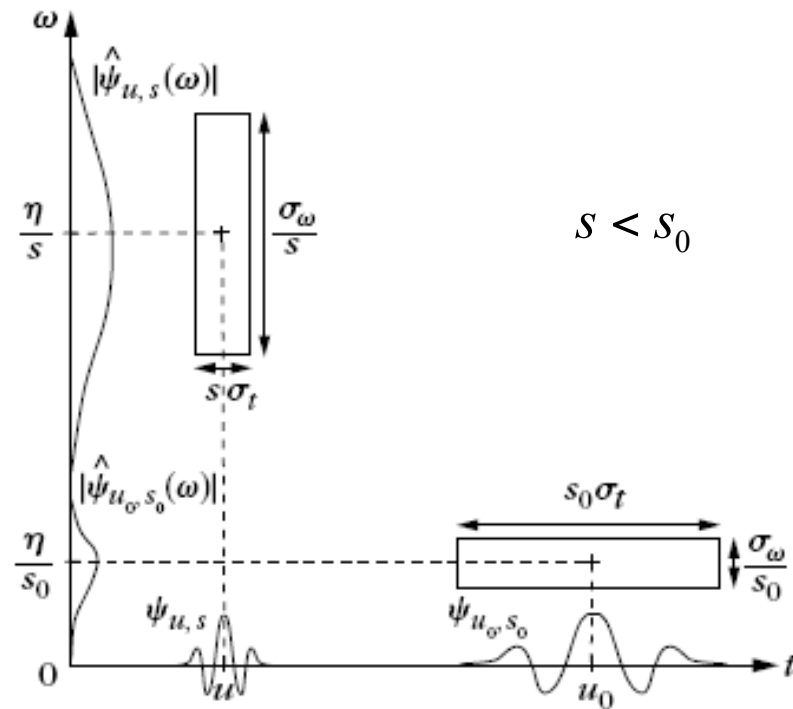
Target

- Non uniform tiling of the time-frequency space
 - This kind of tiling is adapted to analyze the scaling evolution of transients with zooming procedures across scales.



Wavelet basis

- As opposed to windowed Fourier atoms, wavelets have a **time-frequency resolution that changes**.
- The wavelet $\psi_{u,s}$ has a time support centered at u and proportional to s . Let us choose a wavelet whose Fourier transform $\hat{\Psi}(\omega)$ is nonzero in a positive frequency interval centered at η .
- The Fourier transform $\hat{\Psi}_{u,s}(\omega)$ is dilated by $1/s$ and thus is localized in a positive frequency interval centered at η/s ; its size is scaled by $1/s$.



$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) \Leftrightarrow \hat{\Psi}_{u,s}(\omega) = e^{-j\omega u} \sqrt{s} \hat{\Psi}(s\omega)$$

Proof

$$\psi(t) \rightarrow \hat{\psi}(\omega)$$

$$\psi(t-u) \rightarrow e^{-j\omega u} \hat{\psi}(\omega)$$

$$\psi\left(\frac{t-u}{s}\right) = \psi\left(\frac{t'}{s}\right) \quad t' = t-u \rightarrow$$

$$\mathfrak{F}\left\{\psi\left(\frac{t'}{s}\right)\right\} = e^{-j\omega u} \hat{\psi}(\omega) \rightarrow \mathfrak{F}\left\{\psi\left(\frac{t'}{s}\right)\right\} = e^{-j\omega u} |s| \hat{\psi}(s\omega) \rightarrow$$

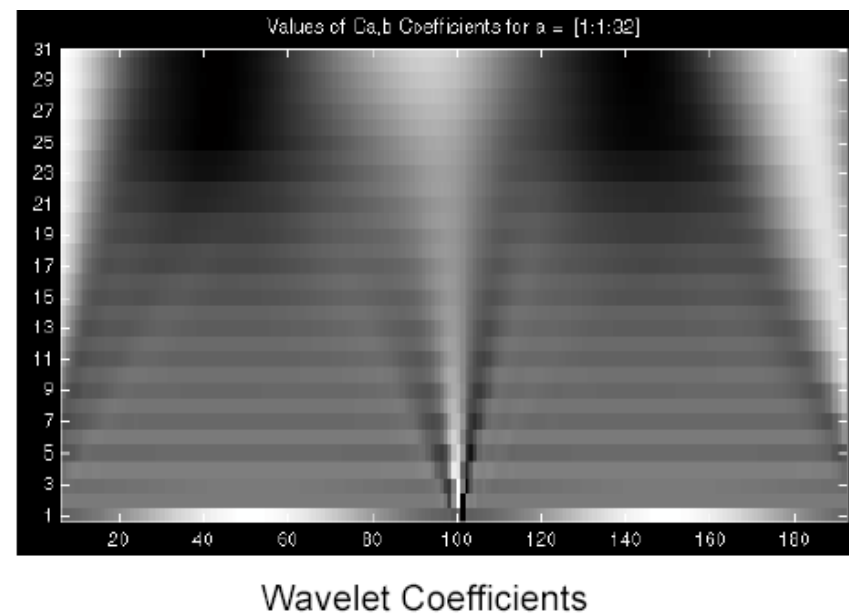
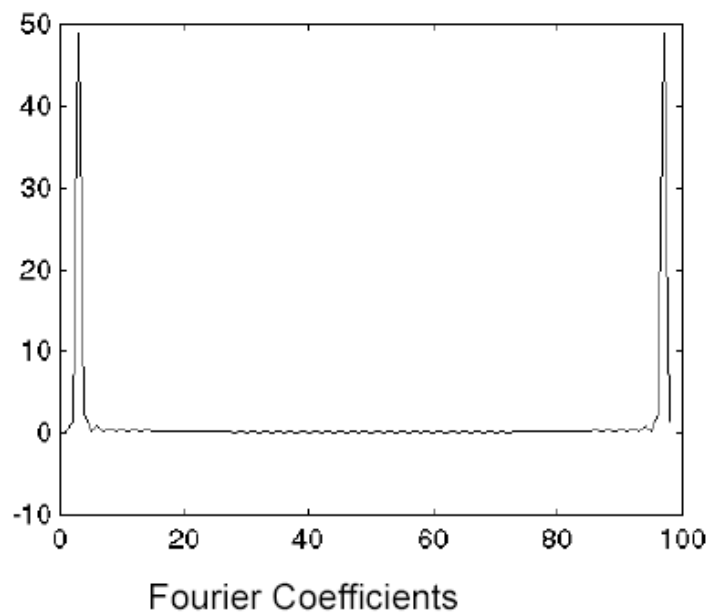
$$\psi\left(\frac{t-u}{s}\right) \leftrightarrow \hat{\psi}(\omega) = e^{-j\omega u} |s| \hat{\psi}(s\omega)$$

Multiscale zooming

- In the time-frequency plane, the Heisenberg box of a wavelet atom u, s is therefore a rectangle centered at $(u, \eta/s)$, with time and frequency widths, respectively, proportional to s and $1/s$.
- When s varies, the time and frequency width of this time-frequency resolution cell changes, but its area remains constant
 - Large-amplitude wavelet coefficients can detect and measure short high frequency variations because they have a narrow time localization at high frequencies.
- At low frequencies their time resolution is lower, but they have a better frequency resolution.
 - This modification of time and frequency resolution is adapted to represent sounds with sharp attacks, or radar signals having a frequency that may vary quickly at high frequencies.

Multiscale zooming

- Signal singularities have specific scaling invariance characterized by Lipschitz exponents.
 - Pointwise regularity of f can be characterized by the asymptotic decay of the wavelet transform amplitude $|Wf(u, s)|$ when s goes to zero.
 - Singularities are detected by following the local maxima of the wavelet transform across scales.

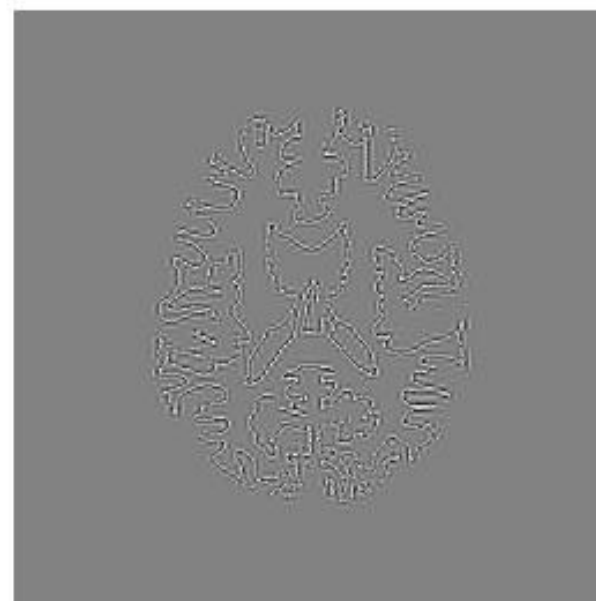


Multiscale zooming

- In images, wavelet local maxima indicate the *position of edges*, which are sharp variations of image intensity.
 - At different scales, the geometry of this local maxima support provides contours of image structures of varying sizes.
 - This multiscale edge detection is particularly effective for pattern recognition in computer vision.



Multiscale edge detection



Sparse representations

Mallat 2009, Chapt. 1

Sparse representations

- Sparse representations: few coefficients reveal the information we are looking for about the signals.
- Such representations can be constructed by decomposing signals over elementary waveforms chosen in a family called a *dictionary*.
 - The discovery of wavelet orthogonal bases and local time-frequency dictionaries has opened the door to a huge jungle of new transforms.
- An *orthogonal basis* is a dictionary of *minimum size* that can yield a sparse representation if designed to concentrate the signal energy over a set of few vectors. This set gives a *geometric* signal description.
- In natural languages, a richer dictionary helps to build shorter and more precise sentences. Similarly, dictionaries of vectors that are *larger* than bases are needed to build sparse representations of complex signals. But choosing is difficult and requires more complex algorithms (*pursuit* algorithms).

Computational harmonic analysis

- Fourier and wavelet bases are the journey's starting point.
- They decompose signals over oscillatory waveforms that reveal many signal properties and provide a path to sparse representations.

Analysis formula → Signal projection on the basis function

Signals → Vectors

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt.$$

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt \Leftrightarrow \int_{-\infty}^{+\infty} f(t) g^*(t) dt = \langle f, g \rangle$$

Fourier → Complex exponentials

SFTF → Windowed complex exponentials (Gabor)

Wavelets → Time and frequency localized scalable functions

Wavelet bases

- Wavelet bases, like Fourier bases, reveal the signal regularity through the amplitude of coefficients, and their structure leads to a fast computational algorithm.
- However, wavelets are well localized and few coefficients are needed to represent local transient structures.
- As opposed to a Fourier basis, a wavelet basis defines a *sparse* representation of piecewise regular signals which may include transients and singularities.
 - In images, large wavelet coefficients are located in the neighborhood of edges and irregular textures.
- First wavelets: Haar
 - Translations and dilations of Haar wavelets generate a basis for $L^2(\mathbb{R})$ for signals having finite energy

$$\psi(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1/2 \\ -1 & \text{if } 1/2 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\|f\|^2 = \int_{-\infty}^{+\infty} |f(t)|^2 dt < +\infty.$$

Wavelet bases

- Writing the projection as an inner product in $L^2(\mathbb{R})$

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} f(t) g^*(t) dt.$$

$$\langle f, \psi_{j,n} \rangle = \int_{-\infty}^{+\infty} f(t) \psi_{j,n}(t) dt \quad \text{Analysis (projection)}$$

$$f = \sum_{j=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \langle f, \psi_{j,n} \rangle \psi_{j,n}. \quad \text{Synthesis (reconstruction)}$$

- Intuition: Each Haar wavelet $\psi_{j,n}(t)$ has a zero average over its support $[2^j n, 2^j(n+1)]$. If f is *locally regular* and 2^j is small, then it is nearly constant over this interval and the wavelet coefficient $\langle f, \psi_{j,n}(t) \rangle$ is nearly zero. This means that *large wavelet coefficients are located at sharp signal transitions only.*

The Wavelets & Filter banks wedding

- Motivated by speech compression, in 1976 Croisier, Esteban, and Galand [189] introduced an invertible filter bank, which decomposes a discrete signal $f[n]$ into two signals of half its size using a filtering and subsampling procedure.
- They showed that $f[n]$ can be recovered from these subsampled signals by canceling the aliasing terms with a particular class of filters called *conjugate mirror filters*.
- The multiresolution theory of Mallat [362] and Meyer [44] proves that **any conjugate mirror filter characterizes a wavelet** that generates an orthonormal basis of $L^2(\mathbb{R})$, and that **a fast discrete wavelet transform is implemented by cascading these conjugate mirror filters** [361].
- The equivalence between this continuous time wavelet theory and discrete filter banks led to a new fruitful interface between digital signal processing and harmonic analysis, first creating a culture shock that is now well resolved.

Continuous vs discrete and finite

- Originally, many signal processing engineers were wondering what is the point of considering wavelets and signals as functions, since all computations are performed over discrete signals with conjugate mirror filters.
- Why bother with the convergence of infinite convolution cascades if in practice we only compute a finite number of convolutions?
- Answering these important questions is necessary in order to understand why this book alternates between theorems on continuous time functions and discrete algorithms applied to finite sequences.
- A short answer would be “simplicity.” In $L^2(\mathbb{R})$, a wavelet basis is constructed by dilating and translating a single function. Several important theorems relate the amplitude of wavelet coefficients to the local regularity of the signal f .
- **Link continuous-discrete:** A theory of continuous-time functions gives *asymptotic* results for discrete sequences with sampling intervals decreasing to zero

Continuous vs discrete and finite

- Continuous time or space models are *not sufficient* for elaborating discrete signal-processing algorithms.
- The transition between continuous and discrete signals must be done with great care to maintain important properties such as orthogonality.
- Restricting the constructions to *finite discrete signals* adds another layer of complexity because of *border problems*.
 - How these border issues affect numerical implementations is carefully addressed once the properties of the bases are thoroughly understood.
- Wavelets for images: wavelet orthonormal bases of images can be constructed from wavelet orthonormal bases of one-dimensional signals (*separability*).
 - This leads to a basis for 2D finite energy functions $f(x_1, x_2)$

Wavelets for images

- Like in one dimension, a wavelet coefficient has a small amplitude if $f(\mathbf{x})$ is regular over the support of $\Psi_{j,n}^k$.
- It has a large amplitude near sharp transitions such as edges.

Sampling with linear approximations

- Analog-to-digital signal conversion is most often implemented with a linear approximation operator that **filters and samples** the input analog signal.
 - From these samples, a linear digital-to-analog converter recovers a projection of the original analog signal over an approximation space whose dimension depends on the sampling density.
- Linear approximations *project* signals in spaces of *lowest* possible dimensions to reduce computations and storage cost, while controlling the resulting error.
- A sampling process implements a filtering of the function $f(x)$ with a low-pass impulse response $\phi_s(x)$ and a uniform sampling to output a discrete signal:

$$f[n] = f * \phi_s(ns) = \int f(x) \phi_s(ns - x) dx = \int f(x) \bar{\phi}_s(x - ns) dx = \langle f(x), \phi_s(ns - x) \rangle$$

$$\phi_s(x) = \bar{\phi}_s(-x).$$

Interpolation

$$\bar{f}_N(x) = \sum_{n=0}^{N-1} f[n] \phi_s(x - ns).$$

Linear approximation error

- The linear approximation error represents the information that is lost in reconstructing the signal using the *first* m basis functions

- Reconstructed signal

$$\bar{f}_N(x) = \sum_{m=0}^{N-1} \langle \bar{f}, \bar{g}_m \rangle \bar{g}_m(x).$$

- Approximation error

$$\varepsilon_l(N, f) = \|\bar{f} - \bar{f}_N\|^2 = \sum_{m=N}^{+\infty} |\langle \bar{f}, \bar{g}_m \rangle|^2.$$

- Exact reconstruction

$$\bar{f} = \sum_{m=0}^{+\infty} \langle \bar{f}, \bar{g}_m \rangle \bar{g}_m.$$

- This error decreases quickly when N increases if the coefficient amplitudes have a fast decay when the index m increases. The dimension N must be adjusted to the desired approximation error.

Linear approximation error

Example: image reconstructed using the first $N/16$ wavelet coefficients at the 3 largest scales



- Reducing the resolution introduces more blur and errors.
- A linear approximation space corresponds to a uniform grid that approximates precisely uniform regular signals.
- Since images are often not uniformly regular, it is necessary to measure it at a high-resolution N .
 - This is why digital cameras have a resolution that increases as technology improves.
- To improve such approximations, more coefficients should be kept where needed—not in regular regions but near sharp transitions and edges.
- *Irregular sampling*

Sparse non linear approximations

- Optimized irregular sampling has a simple equivalent solution through nonlinear approximations in wavelet bases.

- *Nonlinear* approximations operate in two stages.

- First, a linear operator approximates the analog signal f with N samples

$$f[n] = \bar{f} \star \hat{\phi}_s(ns).$$

- Then, a nonlinear approximation of $f[n]$ is computed to reduce the N coefficients $f[n]$ to $M \ll N$ coefficients in a *sparse* representation.

- The discrete signal can be seen as a vector in \mathbb{C}^N where inner products can be written as

$$\langle f, g \rangle = \sum_{n=0}^{N-1} f[n]g^*[n] \quad \text{and} \quad \|f\|^2 = \sum_{n=0}^{N-1} |f[n]|^2.$$

- To obtain a sparse representation with a nonlinear approximation, we choose a new orthonormal basis $\hat{\mathcal{B}} = \{g_m[n]\}_{m \in \Gamma}$ of \mathbb{C}^N which concentrates the signal energy as much as possible over few coefficients

Linear vs Nonlinear approximations

Linear, N/16 *low-scale* coeffs



Non-linear, N/16 *largest* coeffs



Sparsity vs regularity

- Sparse representations are obtained in a basis that takes advantage of some form of *regularity* of the input signals, creating many small-amplitude coefficients.
- Since wavelets have localized support, functions with isolated singularities produce few large-amplitude wavelet coefficients in the neighborhood of these singularities.
- Nonlinear wavelet approximation produces a small error over spaces of functions that do not have “too many” sharp transitions and singularities.
- Note: Edges often define regular geometric curves. Wavelets detect the location of edges but their square support cannot take advantage of their potential geometry regularity. More sparse representations are defined in dictionaries of *curvelets* or *bandlets*, which have elongated support in multiple directions, that can be adapted to this geometrical regularity

Wavelet transforms

Wavelet transforms

- A wavelet is a function of zero average centered in the neighborhood of $t=0$ and is normalized

$$\int_{-\infty}^{+\infty} \psi(t) dt = 0$$
$$\|\psi\| = 1$$

- The translations and dilations of the wavelet generate a family of time-frequency atoms

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right)$$

- Wavelet transform of f in $L^2(\mathbb{R})$ at position u and scale s is

$$Wf(u,s) = \langle f, \psi_{u,s} \rangle = \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{s}} \psi^*\left(\frac{t-u}{s}\right) dt$$

Wavelet transforms

- Real wavelets: suitable for detecting sharp signal transitions

$$Wf(u, s) = \langle f, \psi_{u,s} \rangle = \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) dt$$

- When s goes to zero the decay of the wavelet coefficients characterize the regularity of f in the neighborhood of u
- Edges in images
- Example: Mexican hat (second derivative of a Gaussian)

$$\psi(t) = \frac{2}{\pi^{1/4} \sqrt{3\sigma}} \left(\frac{t^2}{\sigma^2} - 1 \right) \exp\left(-\frac{t^2}{2\sigma^2} \right)$$

$$\hat{\psi}(\omega) = \frac{-\sqrt{8}\sigma^{5/2}\pi^{1/4}}{\sqrt{3}} \omega^2 \exp\left(-\frac{\sigma^2\omega^2}{2} \right)$$

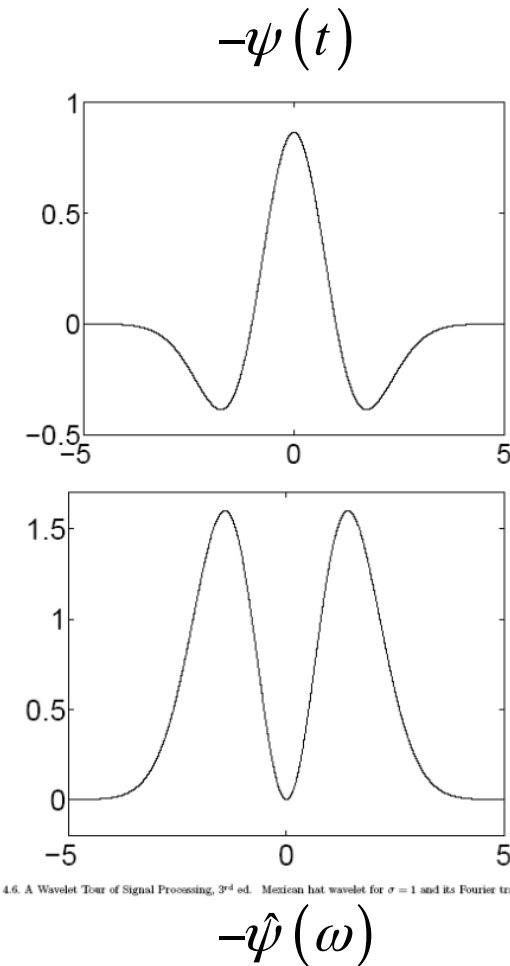
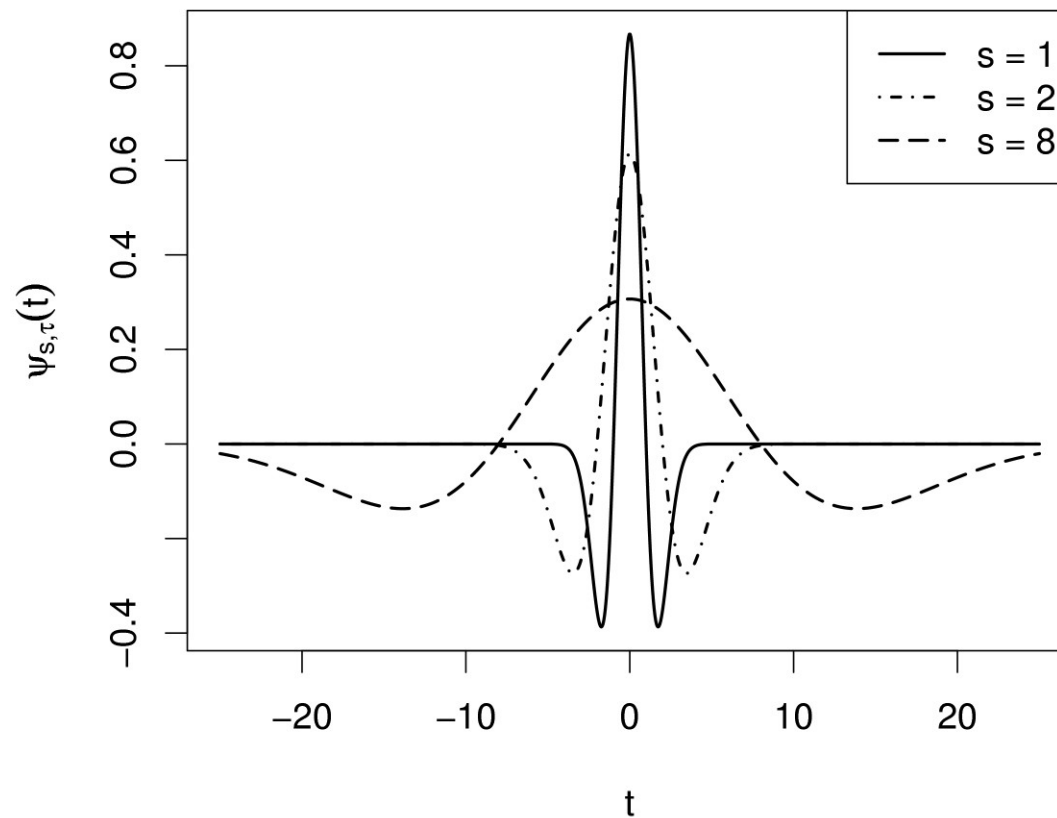


Fig. 4.6. A Wavelet Tour of Signal Processing, 3rd ed. Mexican hat wavelet for $\sigma = 1$ and its Fourier transform.

Real wavelets: example

- The wavelet transform was calculated using a Mexican hat wavelet



Real wavelets: Admissibility condition

- Theorem 4.3 (Calderon, Grossman, Morlet)

Let ψ in $L^2(\mathbb{R})$ be a real function such that

$$C_\psi = \int_0^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < +\infty$$

Admissibility condition

Any f in $L^2(\mathbb{R})$ satisfies

$$f(t) = \frac{1}{C_\psi} \int_0^{+\infty} \int_{-\infty}^{+\infty} Wf(u,s) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) du \frac{ds}{s^2}$$

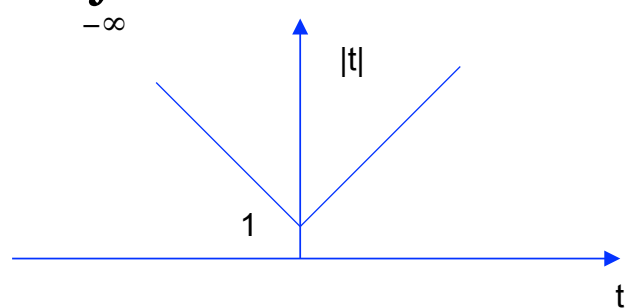
and

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt = \frac{1}{C_\psi} \int_0^{+\infty} \int_{-\infty}^{+\infty} |Wf(u,s)|^2 du \frac{1}{s^2} ds$$

Admissibility condition

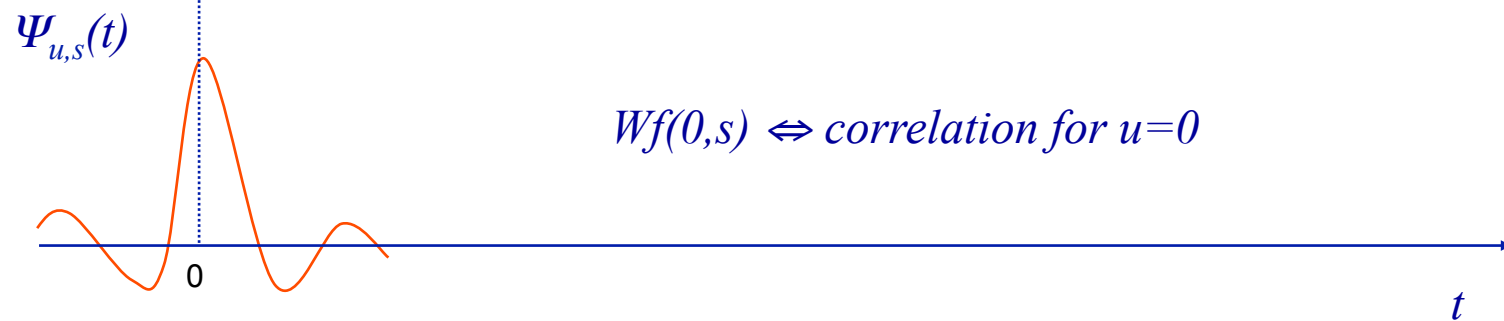
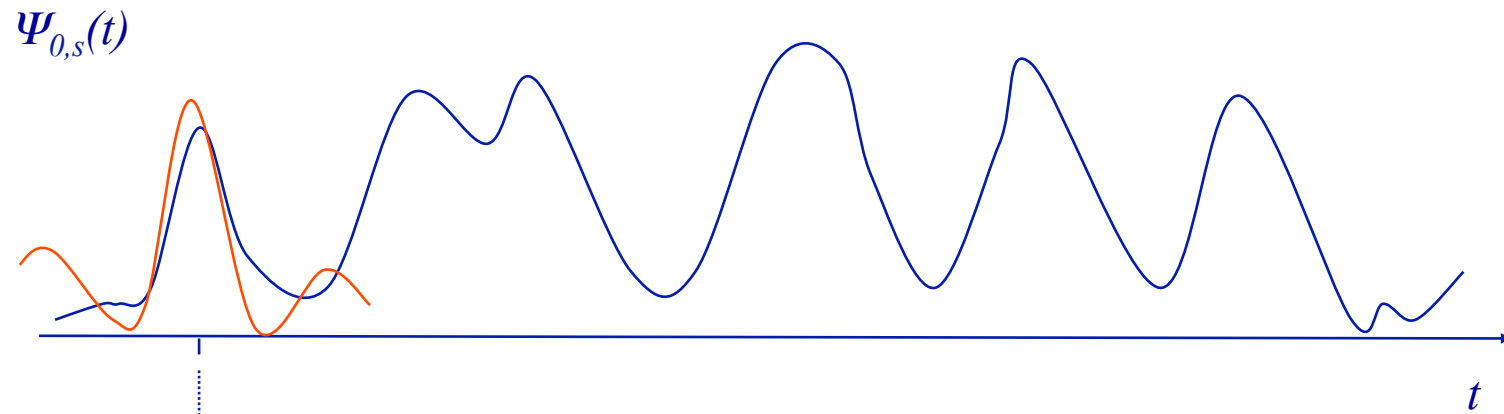
- Consequences

- The integral is finite if the wavelet has zero average $\hat{\psi}(0) = 0$
 - This condition is nearly sufficient \rightarrow
- If $\hat{\psi}(0) = 0$ and $\hat{\psi}(\omega)$ is continuously differentiable, then the admissibility condition is satisfied
 - This happens if it has a sufficient time decay

$$\int_{-\infty}^{+\infty} (1 + |t|) |\psi(t)| dt < +\infty$$


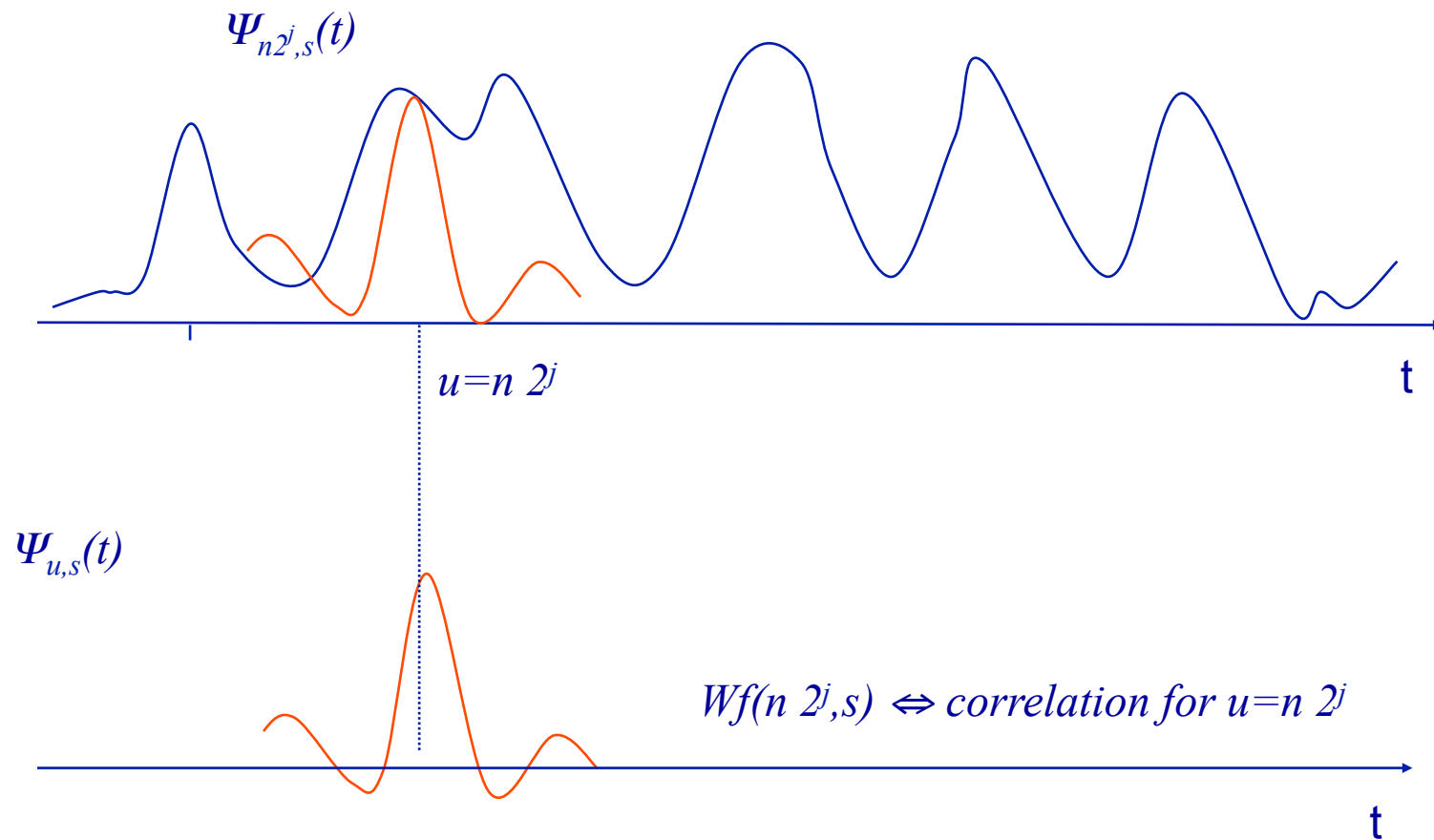
\rightarrow The wavelet function must decay **sufficiently fast** in both time and frequency

Wavelet transform

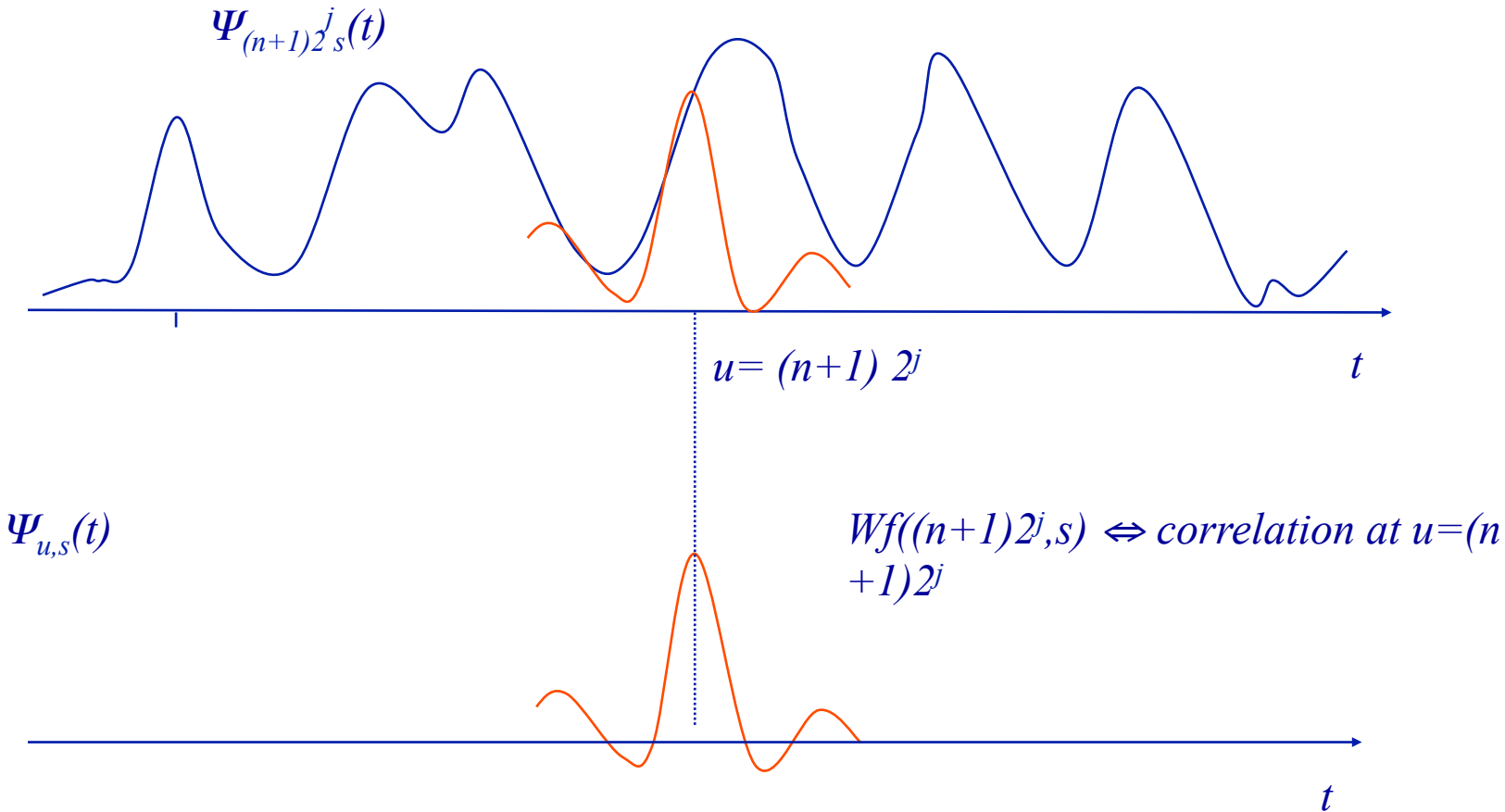


$Wf(0,s) \Leftrightarrow$ correlation for $u=0$

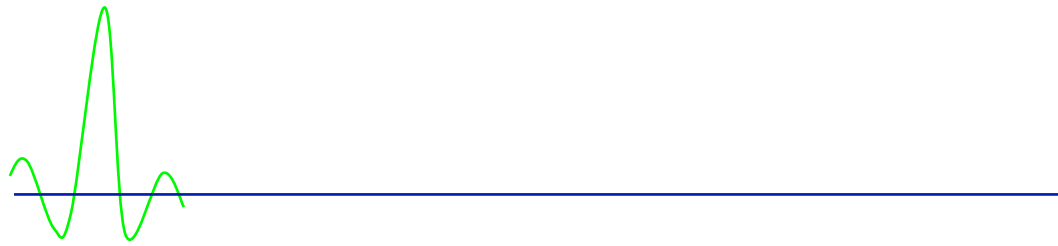
Wavelet transform



Wavelet transform



Changing the scale



$\Psi_{u,s}(t)$

$s=2^{j+1}$

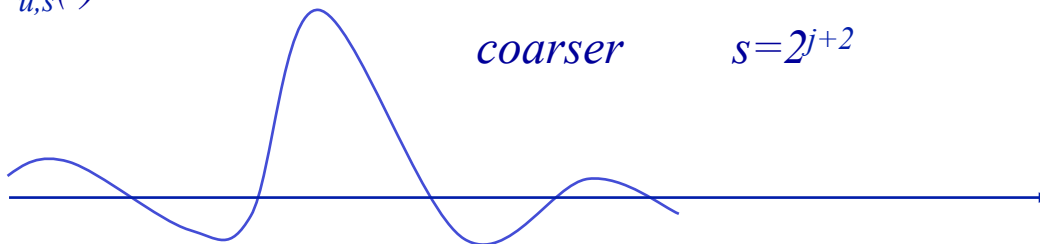
multiresolution



$\Psi_{u,s}(t)$

coarser

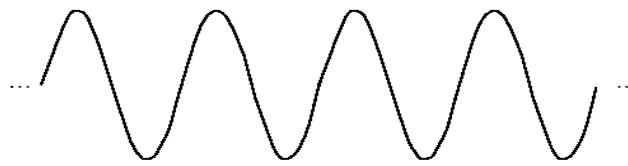
$s=2^{j+2}$



Fourier versus Wavelets

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt$$

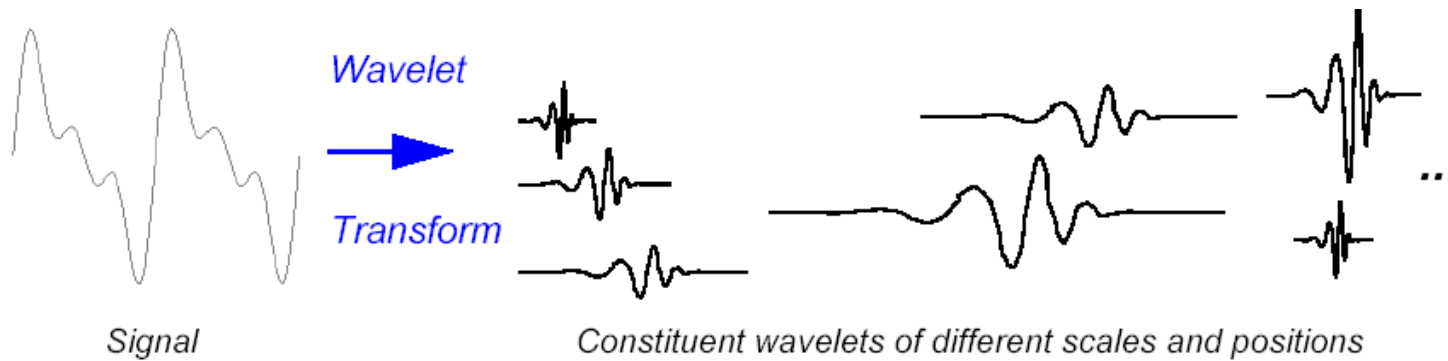
$$C(\text{scale}, \text{position}) = \int_{-\infty}^{\infty} f(t)\psi(\text{scale}, \text{position}, t) dt$$



Sine Wave



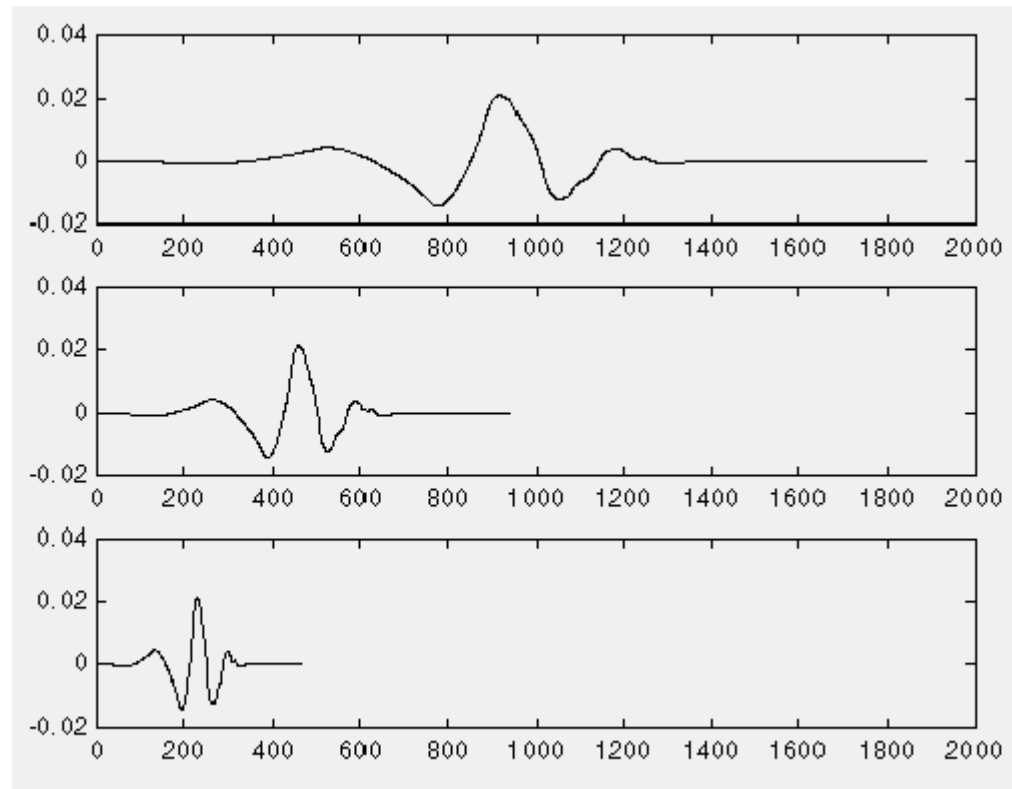
Wavelet (db10)



Signal

Constituent wavelets of different scales and positions

Scaling

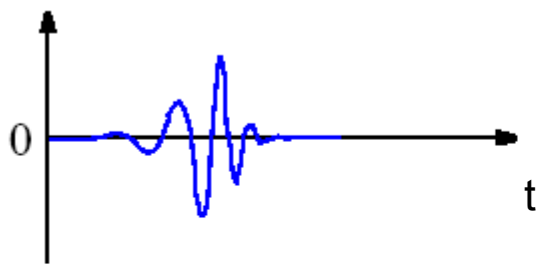


$$f(t) = \psi(t) \quad ; \quad a = 1$$

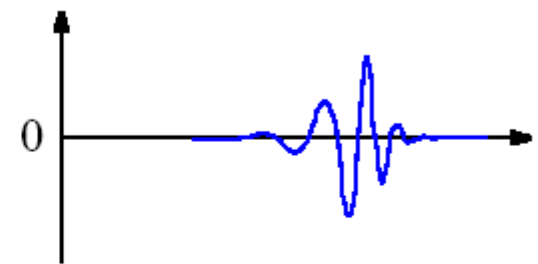
$$f(t) = \psi(2t) \quad ; \quad a = \frac{1}{2}$$

$$f(t) = \psi(4t) \quad ; \quad a = \frac{1}{4}$$

Shifting



Wavelet function
 $\psi(t)$

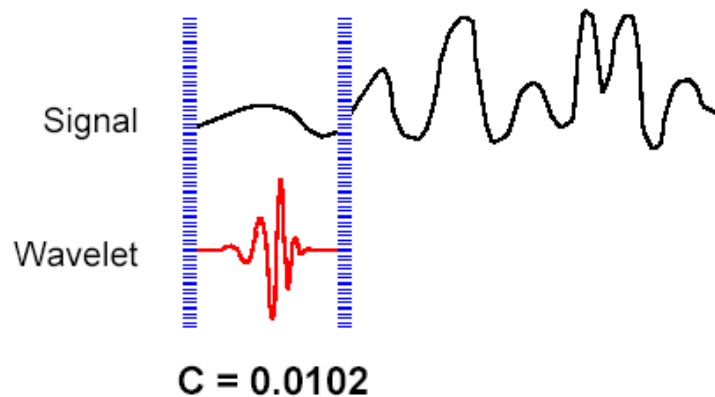


Shifted wavelet function
 $\psi(t-k)$

Recipe

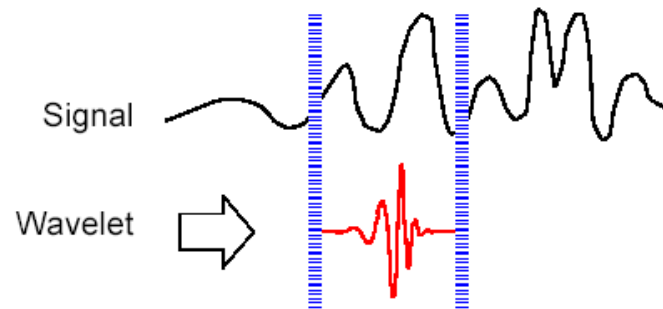
- 1 Take a wavelet and compare it to a section at the start of the original signal.
- 2 Calculate a number, C , that represents how closely correlated the wavelet is with this section of the signal. The higher C is, the more the similarity. More precisely, if the signal energy and the wavelet energy are equal to one, C may be interpreted as a correlation coefficient.

Note that the results will depend on the shape of the wavelet you choose.

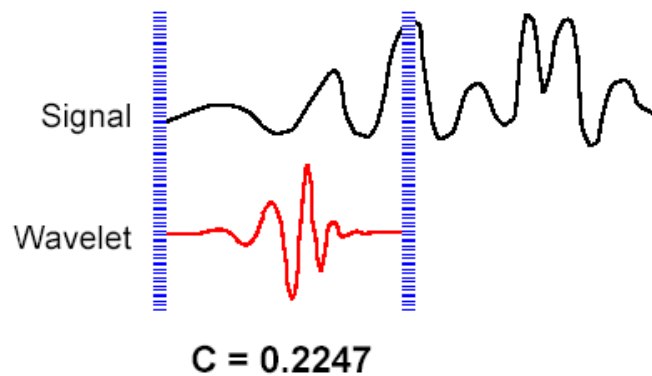


Recipe

- 3 Shift the wavelet to the right and repeat steps 1 and 2 until you've covered the whole signal.



- 4 Scale (stretch) the wavelet and repeat steps 1 through 3.



- 5 Repeat steps 1 through 4 for all scales.

Wavelet Zoom

- WT at position u and scale s measures the local correlation between the signal and the wavelet



Thus, there is a correspondence between wavelet scales and frequency as revealed by wavelet analysis:

- (small) • Low scale $a \Rightarrow$ Compressed wavelet \Rightarrow Rapidly changing details \Rightarrow High frequency ω .
- (large) • High scale $a \Rightarrow$ Stretched wavelet \Rightarrow Slowly changing, coarse features \Rightarrow Low frequency ω .

Frequency domain

- Parseval
$$Wf(u, s) = \int_{-\infty}^{+\infty} f(t) \psi_{u,s}^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) \Psi_{u,s}^*(\omega) d\omega$$

The wavelet coefficients $Wf(u,s)$ depend on the values of $f(t)$ (and $F(\omega)$) in the time-frequency region where the energy of the corresponding wavelet function (respectively, its transform) is concentrated

- time/frequency localization*
- The *position and scale* of high amplitude coefficients allow to characterize the *temporal evolution* of the signal
- Time domain signals (1D) : Temporal evolution
- Spatial domain signals (2D) : Localize and characterize spatial singularities

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) \Leftrightarrow \Psi_{u,s}(\omega) = e^{-j\omega u} \sqrt{s} \Psi(s\omega)$$

Stretching in time \leftrightarrow Shrinking in frequency (and viceversa)

Parseval & Plancherel

Theorem 2.3. If f and h are in $\mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$, then

$$\int_{-\infty}^{+\infty} f(t) h^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega) \hat{h}^*(\omega) d\omega. \quad (2.25)$$

For $h = f$ it follows that

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{f}(\omega)|^2 d\omega. \quad (2.26)$$

Proof. Let $g = f \star \bar{h}$ with $\bar{h}(t) = h^*(-t)$. The convolution, Theorem 2.2, and property (2.23) show that $\hat{g}(\omega) = \hat{f}(\omega) \hat{h}^*(\omega)$. The reconstruction formula (2.8) applied to $g(0)$ yields

$$\int_{-\infty}^{+\infty} f(t) h^*(t) dt = g(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{g}(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega) \hat{h}^*(\omega) d\omega. \quad \blacksquare$$

by definition of convolution

inverse transform in $t=0$

Real valued functions

The FT of a real valued function is symmetric and thus the power spectrum is even

$$f(t) \rightarrow \hat{f}(\omega)$$

$$f(-t) \rightarrow \hat{f}(-\omega) = \hat{f}^*(\omega)$$

Proof

$$\mathfrak{F}\{f(-t)\} = \int_{-\infty}^{+\infty} f(-t) e^{-j\omega t} dt = \int_{-\infty}^{+\infty} f(t') e^{j\omega t'} dt' = \hat{f}(-\omega)$$

Note to Plancherel's formula

Table 2.1 Fourier Transform Properties

Property	Function	Fourier Transform	
	$f(t)$	$\hat{f}(\omega)$	
Inverse	$\hat{f}(t)$	$2\pi f(-\omega)$	(2.15)
Convolution	$f_1 \star f_2(t)$	$\hat{f}_1(\omega)\hat{f}_2(\omega)$	(2.16)
Multiplication	$f_1(t)f_2(t)$	$\frac{1}{2\pi}\hat{f}_1 \star \hat{f}_2(\omega)$	(2.17)
Translation	$f(t-u)$	$e^{-iu\omega}\hat{f}(\omega)$	(2.18)
Modulation	$e^{i\xi t}f(t)$	$\hat{f}(\omega-\xi)$	(2.19)
Scaling	$f(t/s)$	$ s \hat{f}(s\omega)$	(2.20)
Time derivatives	$f^{(p)}(t)$	$(i\omega)^p\hat{f}(\omega)$	(2.21)
Frequency derivatives	$(-it)^p f(t)$	$\hat{f}^{(p)}(\omega)$	(2.22)
Complex conjugate	$f^*(t)$	$\hat{f}^*(-\omega)$	(2.23)
Hermitian symmetry	$f(t) \in \mathbb{R}$	$\hat{f}(-\omega) = \hat{f}^*(\omega)$	(2.24)

Wavelets and linear filtering

- *The WT can be rewritten as a convolution product and thus the transform can be interpreted as a linear filtering operation*

$$Wf(u, s) = \langle f, \psi_{u,s} \rangle = \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{s}} \psi^* \left(\frac{t-u}{s} \right) dt =$$

$$\bar{\psi}_s(t) = \frac{1}{\sqrt{s}} \psi^* \left(\frac{-t}{s} \right)$$

$$\hat{\bar{\psi}}_s(\omega) = \sqrt{s} \hat{\psi}^*(s\omega)$$

$$\hat{\psi}(0) = 0$$

→ *band-pass filter*

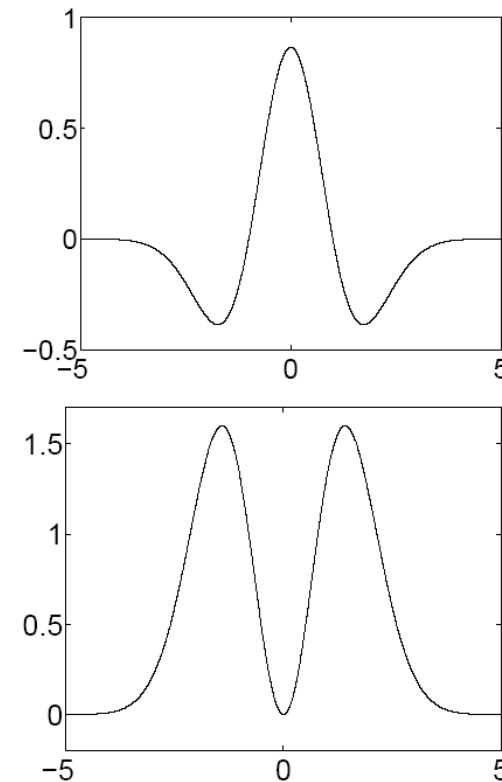
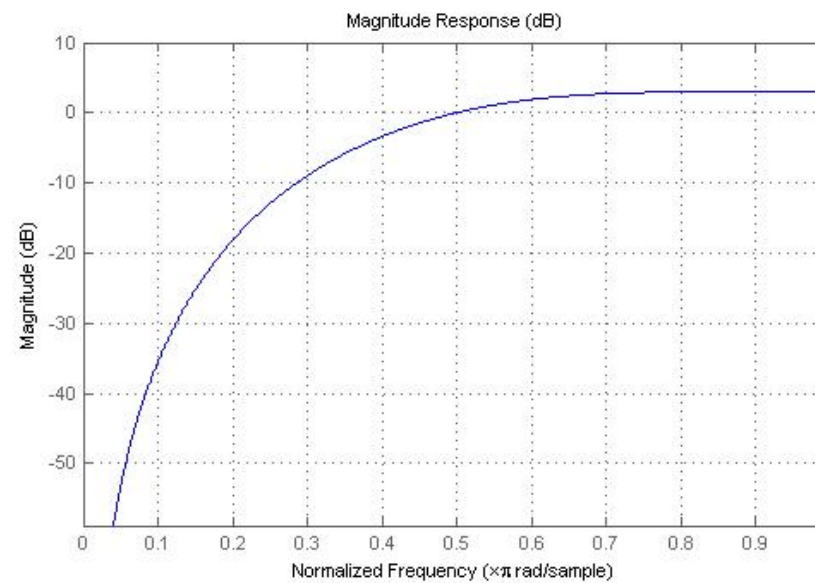
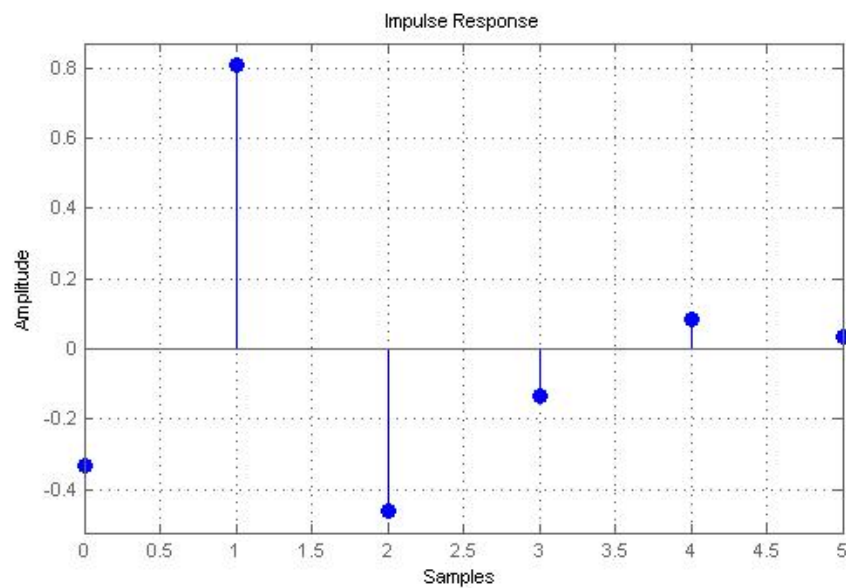
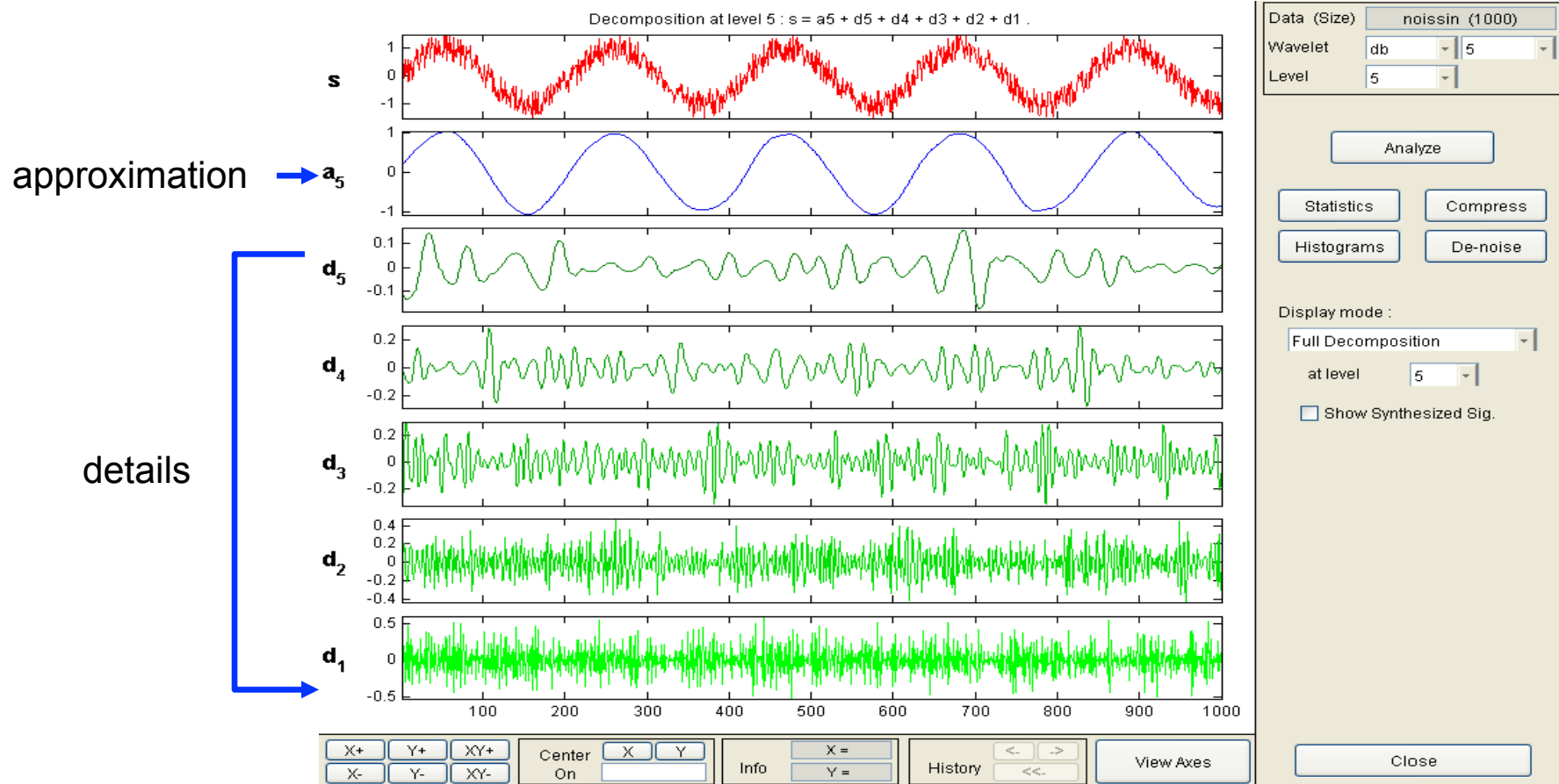


Fig. 4.6. A Wavelet Tour of Signal Processing, 3rd ed. Mexican hat wavelet for $\sigma = 1$ and its Fourier transform.

Wavelet filter (db3)



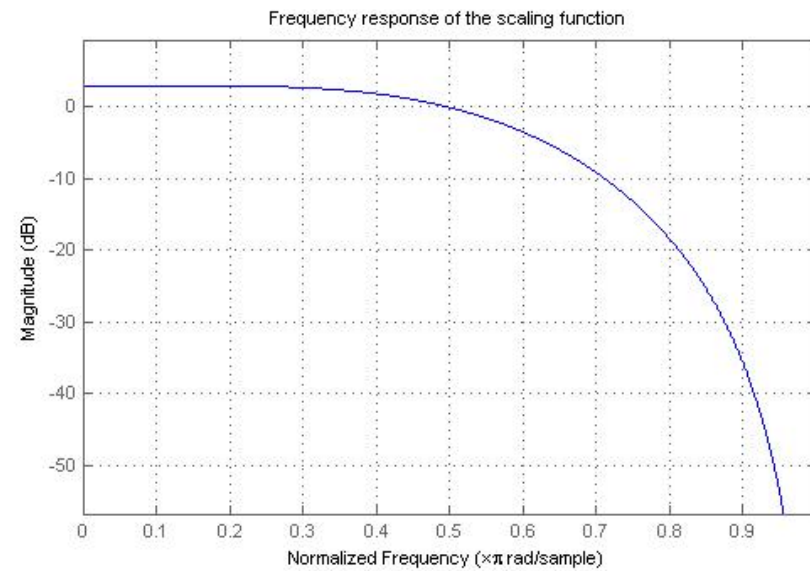
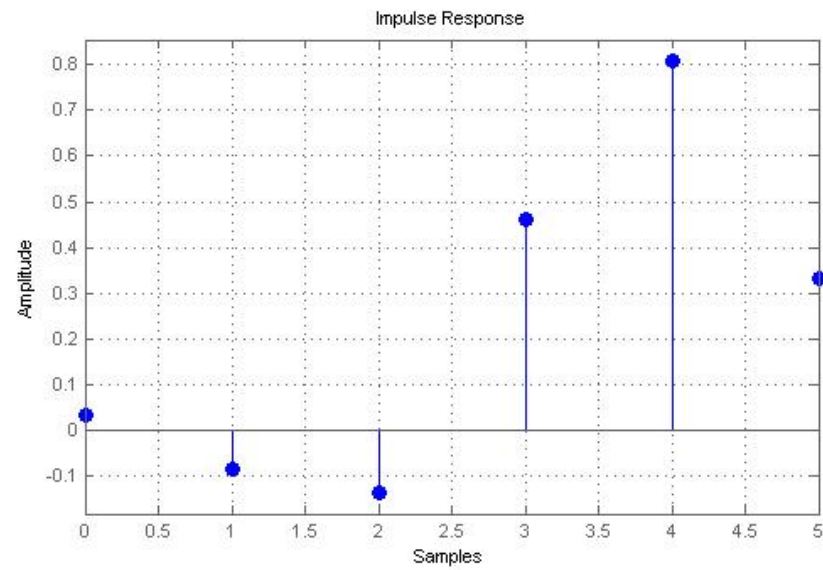
Scaling function



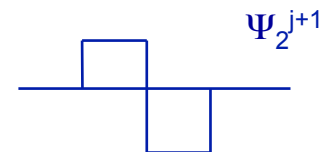
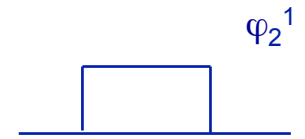
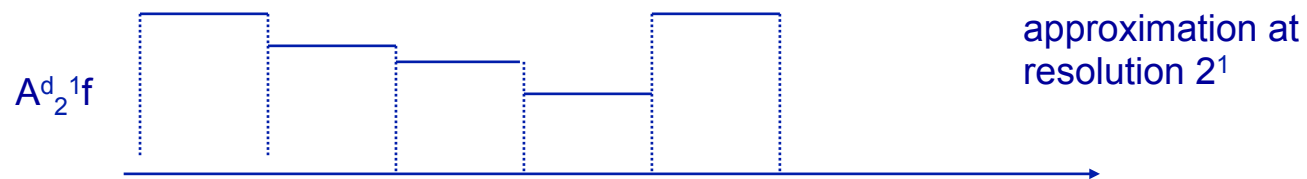
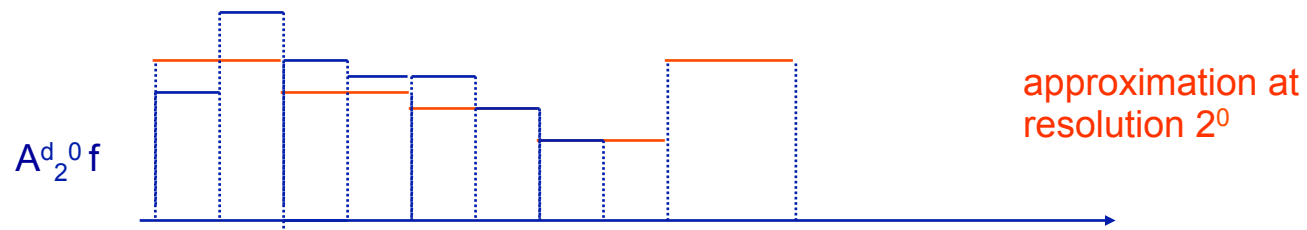
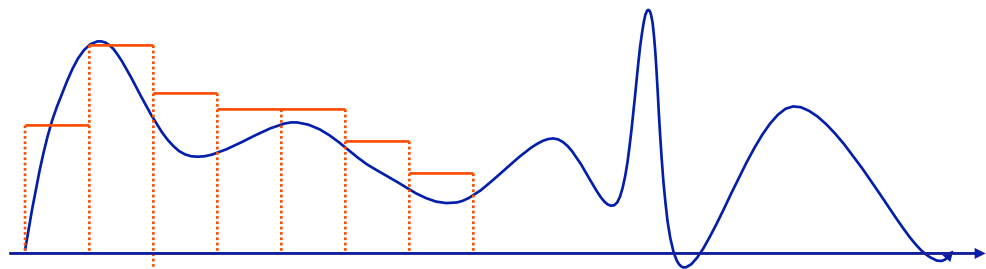
Wavelet representation = approximation + details

approximation ↔ scaling function
 details ↔ wavelets

Scaling function (db3)



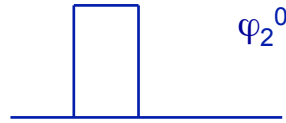
A different perspective



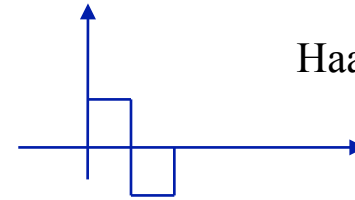
$$A_2^{d,j}f = A_2^{d,j+1}f + d_2^{j+1}f$$

Haar pyramid [Haar 1910]

Haar basis function



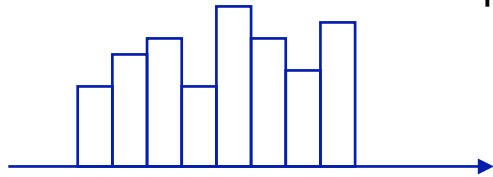
φ_2^0



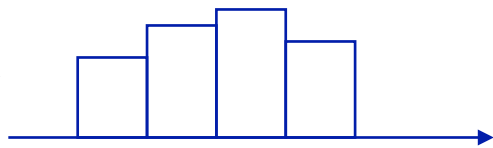
Haar wavelet

reconstructed from discrete approximations

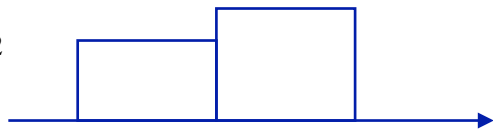
sig₀



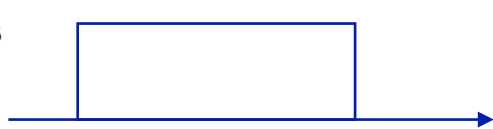
sig₁



sig₂



sig₃

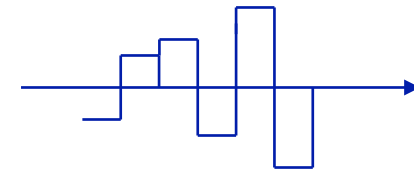


$$\varphi_{i,k} = 2^{-i/2} \varphi(x/2^i - k)$$

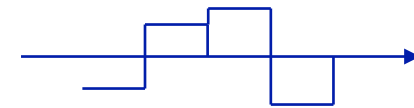
$$\text{sig}_i = \sum_k a_i(k) \varphi_{k,i}$$

residuals from details

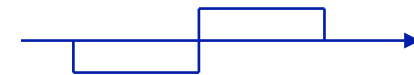
$$r_1 = \sum_k d_1(k) \Psi_{k,1}$$



$$r_2 = \sum_k d_2(k) \Psi_{k,2}$$



$$r_3 = \sum_k d_3(k) \Psi_{k,3}$$



$$s = \text{sig}_3 + \sum_{i,k} d_{k,i}(k) \Psi_{k,i} = \sum_k a_3(k) \varphi_{k,3} + \sum_{i,k} d_{k,i}(k) \Psi_{k,i}$$

Summery history

- Haar wavelet → piece-wise constant functions → far from optimal
- Stronberg 1980 → piece-wise linear functions → better approximation properties
- Meyer 1989 → continuously differentiable functions
- Mallat and Meyer 1989 → Theory for multiresolution signal approximation